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Embedding Classical Indices in the FP Family

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Abstract Recently, a new family of power indices, the *FP*, was introduced by Fragnelli, Ottone and Sattanino (2009), requiring that the parties of a majority are ideologically contiguous along a left-right axis. The different choices of some parameters allow representing various situations, resulting in different indices in this family. In this paper we analyze how to select the parameters with the aim of transferring some properties of classical power indices. We start by relaxing the hypothesis of contiguity. Then, we reduce the relevance of non-contiguous coalitions, defining a sequence of indices that converges to a modified version of the classical indices. The method is applied to the Italian lower chamber. Finally, we extend our approach to situations in which the parties are not necessarily ordered according to the left-right axis, expressing their relations by a graph, following the idea of Myerson (1977).

Keywords Weighted majority games, power indices, contiguous coalitions
JEL classification C70, D72

1. Introduction

In various real-world situations a set of agents has to decide in favor of an issue or against it. Some subsets of agents are able to reach an agreement that makes the issue approved, while some other subsets may at most block it, but they are not able to pass a counterproposal. We may think to an electoral body that has to choose its representatives, or to the parties of a Parliament that have to pass a law, or to a council whose members have to take a decision. A relevant role is played by those agents that mostly can influence the final outcome. Power indices are a tool that allows evaluating the role played by each agent in the process ending with the formation of a majority. Several power indices were proposed in order to account different features of the possible situations; for instance the indices may emphasize the importance of the ordering in the majority formation process (Shapley and Shubik 1954), the possibility to form different majorities (Banzhaf 1965; Coleman 1971), the role played in the majorities with minimal number of agents (Deegan and Packel 1978; Holler 1982).

These indices equally consider all the possible orderings, majority coalitions or minimal winning coalitions. In order to include other information, Owen (1977) introduced the concept of *a priori* unions for accounting existing agreements between parties; Myerson (1977) proposed to use a graph for representing possible connections

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among parties that are the basis for negotiations; we will reconsider the proposal by Myerson in Section 7; Kalai and Samet (1987) allowed the possibility of different roles for the players assigning to each of them a suitable weight. Felsenthal and Machover (1998 and 2005) are two very good surveys on the evaluation of the power in voting situations.

The *FP* family of power indices, introduced in Fragnelli et al. (2009), focuses on the contiguity of the parties of a Parliament ordered on a left-right axis. The basic idea is that a coalition may form after a negotiation that includes all the intermediate parties. A simple example may be given by a party that is never able to change the decision taken by any other coalition (the so-called null player) but may play an important role due to its intermediate position that makes it necessary for a positive conclusion of the negotiation. The indices in the *FP* family depend on the setting of some parameters, namely the set of majorities with contiguous parties that are relevant in the situation at hand, their probabilities to form and the relevance that each member has in each majority.

Taking into account a restricted set of feasible coalitions, the indices of this family allow evaluating the voting power of each party starting from the possible negotiation processes. The resulting power will be influenced by the ideological connections among the parties in a Parliament and not only by the seat share, as it happens with the classical indices.

Exploiting the degrees of freedom of the *FP* family, in this paper we want to select the parameters in order to embed the classical power indices by Shapley-Shubik, Banzhaf-Coleman, Deegan-Packel and Holler, in the new family. The motivation is that the modified indices may profit of some features of the classical ones, adding the relevance assigned to intermediate parties in the new family. Clearly, the characteristic of assigning a null power to a null player, that is satisfied by the four classical indices we mentioned, still holds after the embedding; nevertheless some parties that are relevant in the negotiation process increase their power, while less important parties decrease their own.

The organization of the paper is as follows. In Section 2 we recall the main feature of weighted majority games; Section 3 is dedicated to a short presentation of the classical power indices; in Section 4 the *FP* family is outlined; Section 5 is devoted to the definition of the extended $\overline{FP}$ family and to the formalization of the procedure we used for embedding the classical indices; in Section 6 we apply the procedure to a real situation; Section 7 deals with an extension of our results to a more general case in which we relax the assumption of contiguity of the coalitions; Section 8 concludes.

2. Weighted majority games

Let $N = \{1, 2, \ldots, n\}$ be the non empty finite set of parties of a Parliament. A *vector of weights* $w = (w_1, w_2, \ldots, w_n)$ is associated to N, where w_i, $i \in N$ is a non negative weight given to each party that may represent the percentage of votes, the number of seats and so on. Fixing a *majority quota* q, we obtain a *weighted majority situation* denoted by $[q; w_1, w_2, \ldots, w_n]$. Given a weighted majority situation it is possible to define the corresponding *weighted majority game* (N,w), where N is the set of players and $w : 2^N \to \{0,1\}$ is the characteristic function defined as

$$w(S) = \begin{cases} 1 & \text{if } \sum_{j\in S} w_j \geq q \\ 0 & \text{otherwise} \end{cases} \qquad \forall S \subseteq N.$$

In such a game we refer to a coalition S with $w(S) = 1$ as a *winning coalition*, i.e. a set of parties which is able to reach the majority quota summing up the weights of all the members of the coalition. The coalition is called *losing* otherwise. Given a winning coalition S, a party $j \in S$ is *critical* for S if $S \setminus \{j\}$ is losing. The quantity $w(S) - w(S \setminus \{j\})$ is called the *marginal contribution of player j w.r.t. S*. We say a winning coalition is *minimal* if each proper subcoalition is losing.

A game (N,w) is called *monotonic* if $S \subseteq T \Rightarrow w(S) \leq w(T)$; a *simple* game is a monotonic game with the condition that $w(S) = 0$ or 1 for each $S \subset N$ and $w(N) = 1$. A simple game is *proper* if $w(S) = 1$ implies $w(N \setminus S) = 0$ for each $S \subseteq N$. A weighted majority game (N,w) results to be monotonic, simple and with the condition $q > 1/2 \sum_{i\in N} w_i$ it is also proper; when the weights represent the number of seats of the parties the condition may be written as $q \geq \lfloor (w_1 + w_2 + \ldots + w_n)/2 + 1 \rfloor$ where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

3. Some classical power indices

Given a game $G = (N,w)$, with $N = \{1,2,\ldots,n\}$, an *allocation* is an n-dimensional real vector $x = (x_1, x_2, \ldots, x_n)$ which assigns the amount x_i to player $i \in N$. An allocation is *efficient* if $x(N) = \sum_{i\in N} x_i = w(N)$. A *solution* is a function ψ that assigns an efficient allocation $\psi(w)$ to every game (N,w). For a simple game, we refer to a non negative solution $\psi(w)$ as a *power index*, where $\psi_i(w)$ represents the power assigned to the player $i \in N$.

A power index for a game (N,w) based on a weighted sum of the marginal contributions of each player is defined as

$$\psi_j(w) = \sum_{S\subseteq N, S\ni j} p(S)[w(S) - w(S \setminus \{j\})] \quad \forall j \in N$$

or, denoting by W the set of winning coalitions, as

$$\psi_j(w) = \sum_{S\in W, S\ni j} p(S)[w(S) - w(S \setminus \{j\})] \quad \forall j \in N,$$

because for each losing coalition S we have $w(S) - w(S \setminus \{j\}) = 0$ for each $j \in S$.

Several power indices based on the marginal contribution have been introduced; the most common are:

(i) the *Shapley-Shubik index* (Shapley and Shubik 1954), given by the following formula

$$\phi_j(w) = \sum_{S\in W, S\ni j} \frac{(|S|-1)!(n-|S|)!}{n!}[w(S) - w(S \setminus \{j\})], \tag{1}$$

where $|S|$ is the cardinality of the set S;

(ii) the *normalized Banzhaf-Coleman index* (Banzhaf 1965; Coleman 1971), defined as

$$\beta_j(w) = \frac{\beta_j^*(w)}{\sum_{k\in N} \beta_k^*(w)} \quad \forall j \in N, \tag{2}$$

where $\beta_j^*(w) = \sum_{S\in W, S\ni j} 1/2^{n-1}[w(S) - w(S \setminus \{j\})]$ for each $j \in N$.

There exist other power indices not based on the marginal contribution which are also common in literature, as the Deegan-Packel index and the Holler index which consider only the set of minimal winning coalitions W^m. The *Deegan-Packel index* (Deegan and Packel 1978), $\delta(w)$, equally divides the power among the coalitions in W^m and then the power of each coalition is equally shared among its members. The *Holler index* (Holler 1982), or *Public Goods index*, $H(w)$, divides the power proportionally to the number of minimal winning coalitions which a player belongs to. Formally we have

$$\delta_j(w) = \sum_{S\in W^m, S\ni j} \frac{1}{|W^m|}\frac{1}{|S|} \quad \forall j \in N \tag{3}$$

and

$$H_j(w) = \frac{h_j}{\sum_{k\in N} h_k} \quad \forall j \in N, \tag{4}$$

where h_j is the number of minimal winning coalitions including player $j \in N$.

4. The *FP* family

The ideological position of each party does not allow every coalition forming with the same probability, even if it is winning. A common scheme to describe a political scenario is a *left-right axis* where the parties in the set N are ordered according to their ideological position. Usually, the axis is represented by the segment 0-1 and the locations of the parties represent their ideology, where 0 is the extreme left and 1 is the extreme right. Assuming that the negotiations take place uniquely between adjacent parties, the feasible coalitions include only contiguous parties. Let W^c be the set of *contiguous winning coalitions*, where a coalition $S_i \in W^c$ is *contiguous* if for all $k, h \in S_i$ if there exists $j \in N$ with $k < j < h$ then $j \in S_i$.

Starting from this idea a new family of power indices, the *FP* family, was defined (Fragnelli et al. 2009). The general formula of an *FP index* is

$$FP_j = \sum_{S_i\in W^c, S_i\ni j} \alpha_i \beta_{ij} \quad \forall j \in N, \tag{5}$$

where $\alpha_i \geq 0$ represents the relative probability of coalition S_i to form, with the condition

$$\sum_{S_i\in W^c} \alpha_i = 1 \tag{6}$$

and $\beta_{ij} \geq 0$ is the power share assigned to player j in S_i, with the condition

$$\sum_{j \in S_i} \beta_{ij} = 1 \quad \forall S_i \in W^c.$$

The choice of parameters α_i differentiates the power of the coalitions and the choice of parameters β_{ij} differentiates the role of the parties inside coalitions. We address to Fragnelli et al. (2009) for possible methods to compute the values of the parameters α_i and β_{ij} that account for the ideological distances, number of parties in the majority, their number of seats or via a suitable analysis of historical data.

We can notice that only contiguous winning coalitions are given a probability to form. In particular we remark that the definition of the *FP* family allows considering even a subset of contiguous winning coalitions, but this is equivalent to assigning a null probability to the remaining contiguous coalitions.

5. Embedding classical indices

For a weighted majority game, classical indices do not take into account only the contiguous coalitions. In order to embed them in the *FP* family, we need an extension of the formula (5) which allows the winning but non-contiguous coalitions to have a probability to form. We define the extended family $\overline{FP}$ as

$$\overline{FP}_j = \sum_{S_i \in W, S_i \ni j} \alpha_i \beta_{ij} \quad \forall j \in N, \tag{7}$$

where $\alpha_i \geq 0$ and $\beta_{ij} \geq 0$ have the same interpretation as above, with the conditions

$$\sum_{S_i \in W} \alpha_i = 1 \tag{8}$$

and

$$\sum_{j \in S_i} \beta_{ij} = 1 \quad \forall S_i \in W. \tag{9}$$

In a following step we look for a standard *FP* index, summing only on the contiguous winning coalitions.

Given a generic index ψ, we want to impose the relations

$$\sum_{S_i \in W, S_i \ni j} \alpha_i \beta_{ij} = \psi_j(w) \quad \forall j \in N \tag{10}$$

via a suitable choice of parameters.

In particular we will analyze the Shapley-Shubik, Banzhaf-Coleman, Deegan-Packel and Holler indices. We can observe there are several choices of the parameters in order to satisfy relations (10) as the system is overdetermined. For instance, a trivial solution is given by $\alpha_N = 1$ for the grand coalition and zero for the others α_i, $i \neq N$ and $\beta_{Nj} = \psi_j$. This solution is not very interesting as it allows only the grand coalition forming and it assumes we already know the value of the index in order to evaluate the parameters β_{ij}. We look now for a non trivial solution, at first for the Shapley-Shubik index and then for the other indices.

We just want to remember that the final aim of the paper is to combine the issue of contiguity with the philosophy of the indices, as the marginal contribution for the Shapley-Shubik and the Banzhaf-Coleman indices and the minimal winning coalitions for the Holler and the Deegan-Packel indices. After defining the appropriate parameters to write as $\overline{FP}$ indices the classical ones, we introduce the idea of contiguity.

5.1 The Shapley-Shubik index

Following the purpose of embedding classical indices in the FP family, we start from the most common one, the Shapley-Shubik index. In particular, the aim is to determine suitable values for the parameters in (7) in order to describe the formula given in (1). We start by imposing that

$$\alpha_i\beta_{ij} = p(S_i)[w(S_i) - w(S_i \setminus \{j\})] \quad \forall S_i \in W, \forall j \in S_i. \tag{11}$$

Summing on $j \in S_i$ and because of the condition (9) this is equal to

$$\alpha_i = p(S_i) \sum_{j\in S_i} [w(S_i) - w(S_i \setminus \{j\})] \quad \forall S_i \in W.$$

Let S_i^c be the set of the critical players of S_i and $c_i = |S_i^c|$. Observe that if player j is critical for S_i then $w(S_i) - w(S_i \setminus \{j\}) = 1$, otherwise $w(S_i) - w(S_i \setminus \{j\}) = 0$. We can write the parameters α_i as

$$\alpha_i = p(S_i)c_i \quad \forall S_i \in W. \tag{12}$$

Condition (8) holds if

$$\sum_{S_i\in W} p(S_i)c_i = \sum_{S_i\in W} \sum_{j\in S_i} p(S_i)[w(S_i) - w(S_i \setminus \{j\})] = 1,$$

which is obviously true. By relations (11) and (12) we obtain

$$\beta_{ij}p(S_i)c_i = p(S_i)[w(S_i) - w(S_i \setminus \{j\})] \quad \forall S_i \in W, \forall j \in S_i,$$

from which we get

$$\beta_{ij} = \begin{cases} \frac{1}{c_i} & \text{if } j \in S_i^c \\ 0 & \text{otherwise} \end{cases} \quad \forall S_i \in W. \tag{13}$$

Trivially, condition (9) is satisfied.

Relations (12) and (13) provide the parameters α_i and β_{ij}, respectively, that enable us to write the Shapley-Shubik index as an $\overline{FP}$ index.

In the definition of the $\overline{FP}$ indices family, the contiguity of the parties is not important. We want now to come back to the idea of contiguous coalitions as the only alliances which are allowed forming, so the power of a party depends only on the coalitions in W^c it belongs to. In order to obtain an FP index, we decrease the probability to form of the non-contiguous coalitions modifying the parameters α_i given in (12). For each coalition $S_i \in W$ we introduce a sequence of parameters $((\gamma_i)_t)_{t\in\mathbb{N}}$ defined as

$$(\gamma_i)_t = \begin{cases} p(S_i)c_i & \text{if } S_i \in W^c \\ (p(S_i)c_i)^t & \text{if } S_i \in W \setminus W^c, \end{cases}$$

from which we get a sequence of normalized parameters $((\alpha_i)_t)_{t\in\mathbb{N}}$

$$(\alpha_i)_t = \frac{(\gamma_i)_t}{\sum_{S_k\in W}(\gamma_k)_t} \quad \forall S_i \in W.$$

It is easy to check that $(\alpha_i)_1 = \alpha_i$ for each $S_i \in W$.

In order to decrease the probability of non-contiguous coalitions to form, we take the limit for $t \to +\infty$ which gives us

$$(\gamma_i)_t \to \gamma_i^*,$$

where

$$\gamma_i^* = \begin{cases} p(S_i)c_i & \text{if } S_i \in W^c \\ 0 & \text{if } S_i \in W \setminus W^c \end{cases}$$

as $p(S_i)c_i < 1$ if $S_i \neq N$, while $N \in W^c$.

Consequently $(\alpha_i)_t$ converges to $\alpha_i^* = \gamma_i^* / \sum_{S_k\in W} \gamma_k^*$ that, using the data of the problem, can be written as

$$\alpha_i^* = \begin{cases} \dfrac{p(S_i)c_i}{\sum_{S_k\in W^c} p(S_k)c_k} & \text{if } S_i \in W^c \\ 0 & \text{if } S_i \in W \setminus W^c. \end{cases} \tag{14}$$

These values respect condition (6) by definition. Note that the sum in (14) does not consider the values $p(S_k)c_k$ for non-contiguous coalitions for which $\gamma_k^* = 0$.

We can assume the definition of β_{ij} does not depend on t, so $\beta_{ij}^* = (\beta_{ij})_t = \beta_{ij}$ for each $t \geq 1$.

The values of parameters α_i^* and β_{ij}^* allow us embedding the Shapley-Shubik index in the *FP* family defining a new index ϕ^{FP}

$$\phi_j^{FP}(w) = \sum_{S_i\in W^c, S_i^c \ni j} \left(\frac{p(S_i)c_i}{\sum_{S_k\in W^c} p(S_k)c_k} \frac{1}{c_i} \right) \quad \forall j \in N. \tag{15}$$

5.2 Other solutions

The procedure we used for the Shapley-Shubik index may be applied to any power index in the family $\overline{FP}$. Let us assume we have an $\overline{FP}$ index given by (7) which respects the conditions (8) and (9), with the additional hypotheses that $\alpha_i < 1$ for each non-contiguous coalition $S_i \in W \setminus W^c$ and $\alpha_i > 0$ for at least one contiguous coalition $S_i \in W^c$. We may decrease the probability of the non-contiguous coalitions to form by defining

$$(\gamma_i)_t = \begin{cases} \alpha_i & \text{if } S_i \in W^c \\ (\alpha_i)^t & \text{if } S_i \in W \setminus W^c \end{cases}$$

and

$$(\alpha_i)_t = \frac{(\gamma_i)_t}{\sum_{S_k \in W} (\gamma_k)_t} \quad \forall S_i \in W, \tag{16}$$

whose limit value is

$$\alpha_i^* = \begin{cases} \dfrac{\alpha_i}{\sum_{S_k \in W^c} \alpha_k} & \text{if } S_i \in W^c \\ 0 & \text{if } S_i \in W \setminus W^c. \end{cases} \tag{17}$$

Again, we assume that the parameters β_{ij} do not depend on t, i.e. $\beta_{ij}^* = (\beta_{ij})_t = \beta_{ij}$ for each $t \geq 1$.

We notice that for each positive $t \in \mathbb{N}_>$ the vector $(\overline{FP})_t$, defined as $(\overline{FP}_j)_t = \sum_{S_i \in W} ((\alpha_i)_t \beta_{ij})$, is a power index that assigns a reduced probability to form to the non-contiguous winning coalitions, as stated in the following proposition.

Proposition 1. *For each power index $\overline{FP}$ and for each $t \in \mathbb{N}_>$ we have that $(\overline{FP})_t = ((\overline{FP}_1)_t, \ldots, (\overline{FP}_n)_t)$ is a power index, i.e. $(\overline{FP}_j)_t \geq 0 \; \forall j \in N$ and $\sum_{j \in N} (\overline{FP}_j)_t = 1$.*

Proof. $(\overline{FP}_j)_t \geq 0$ for each $j \in N$ and for each $t \in \mathbb{N}_>$ by definition. The value of $(\overline{FP}_j)_t$ for each $t \in \mathbb{N}_>$ is

$$\begin{aligned} (\overline{FP}_j)_t &= \sum_{S_i \in W} ((\alpha_i)_t \beta_{ij}) \\ &= \sum_{S_i \in W^c} \left(\frac{(\gamma_i)_t}{\sum_{S_k \in W} (\gamma_k)_t} \beta_{ij} \right) + \sum_{S_i \in W \setminus W^c} \left(\frac{(\gamma_i)_t}{\sum_{S_k \in W} (\gamma_k)_t} \beta_{ij} \right). \end{aligned}$$

So

$$\begin{aligned} \textstyle\sum_{j \in N} (\overline{FP}_j)_t &= \sum_{j \in N} \sum_{S_i \in W^c} \left(\frac{(\gamma_i)_t}{\sum_{S_k \in W} (\gamma_k)_t} \beta_{ij} \right) + \sum_{j \in N} \sum_{S_i \in W \setminus W^c} \left(\frac{(\gamma_i)_t}{\sum_{S_k \in W} (\gamma_k)_t} \beta_{ij} \right) \\ &= \sum_{S_i \in W^c} \sum_{j \in N} \left(\frac{(\gamma_i)_t}{\sum_{S_k \in W} (\gamma_k)_t} \beta_{ij} \right) + \sum_{S_i \in W \setminus W^c} \sum_{j \in N} \left(\frac{(\gamma_i)_t}{\sum_{S_k \in W} (\gamma_k)_t} \beta_{ij} \right) \\ &= \frac{\sum_{S_i \in W^c} (\gamma_i)_t}{\sum_{S_k \in W} (\gamma_k)_t} \sum_{j \in N} \beta_{ij} + \frac{\sum_{S_i \in W \setminus W^c} (\gamma_i)_t}{\sum_{S_k \in W} (\gamma_k)_t} \sum_{j \in N} \beta_{ij} \\ &= \frac{\sum_{S_i \in W} (\gamma_i)_t}{\sum_{S_k \in W} (\gamma_k)_t} = 1. \end{aligned}$$

□

To embed the other classical power indices of Banzhaf-Coleman, Deegan-Packel and Holler in the $\overline{FP}$ family and, consequently, to obtain the corresponding FP indices, is now simply a matter of suitably defining the parameters α_i and β_{ij}.

Table 1. Parameters to embed some classical indices in the $\overline{FP}$ family

Parameters	α_i	β_{ij}
Banzhaf-Coleman	$\frac{c_i}{\sum_{S_k \in W} c_k}, \quad S_i \in W$	$\begin{cases} \frac{1}{c_i} & \text{if } j \in S_i^c \\ 0 & \text{otherwise} \end{cases}$
Deegan-Packel	$\begin{cases} \frac{1}{\lvert W^m \rvert} & \text{if } S_i \in W^m \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} \frac{1}{c_i} & \text{if } j \in S_i^c \\ 0 & \text{otherwise} \end{cases}$
Holler	$\begin{cases} \frac{c_i}{\sum_{S_k \in W^m} c_k} & \text{if } S_i \in W^m \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} \frac{1}{c_i} & \text{if } j \in S_i^c \\ 0 & \text{otherwise} \end{cases}$

For the normalized Banzhaf-Coleman index, the probability of a coalition to form is proportional to the number of critical players. The same holds for the Holler index, with the difference that non minimal coalitions have null probability to make an agreement and for each minimal one the number of critical players is equal to the cardinality of the coalition itself. Differently, the Deegan-Packel index takes into account only minimal coalitions but it assumes they have all the same probability to create an agreement. The sharing of the power between players inside a given winning coalition is always given by an equal division between critical players.

In Table 1 we summarize the choice of the parameters for the Banzhaf-Coleman, the Deegan-Packel and the Holler indices, respectively, in order to write them as $\overline{FP}$ indices.

By the procedure presented in Section 5.1 we obtain the following formulas. For the normalized Banzhaf-Coleman index we get

$$\beta_j^{FP}(w) = \frac{h_j^c}{\sum_{k \in N} h_k^c} \quad \forall j \in N, \tag{18}$$

where h_j^c counts how many times a player is critical in a contiguous winning coalition.

Embedding the Deegan-Packel index in the *FP* family, we obtain an index where the probability to form is the same for all the coalitions in the set W^{mc} of contiguous minimal winning coalitions and the power of each coalition is equally divided among its members

$$\delta_j^{FP}(w) = \sum_{S_i \in W^{mc}} \frac{1}{|W^{mc}|} \frac{1}{|S_i|} \quad \forall j \in N. \tag{19}$$

Note that this index coincides with the archetype of the *FP* family (see Fragnelli et al. 2009). Finally, the Holler index adapted to the *FP* family is given by

$$H_j^{FP}(w) = \frac{h_j^{mc}}{\sum_{k \in N} h_k^{mc}} \quad \forall j \in N, \tag{20}$$

where h_j^{mc} counts how many times a player belongs to a contiguous minimal winning coalition.

6. An example

In this section we apply the results presented in the previous sections to a real Parliament, the Italian lower chamber, *Camera dei Deputati* (Chamber of Deputies or Camera), that is formed by 630 members, whose majority quota is $\lfloor v/2+1 \rfloor$, where v is the number of voters, excluding absences and abstentions. The required quorum during a legislative vote is 316 Deputies. The data used in the example and shown in Table 2 are taken from the general election of April 2008; for sake of simplicity we decided not to consider 18 seats belonging to very small parties which, historically, have no practical influence on the decisions of the Camera even if, in theory, they could change the outcome. The remaining 612 seats are assigned as in Table 2 to five parties, from left to right, Italia dei Valori (IdV, Italy for Values), Partito Democratico (PD, Democratic Party), Unione di Centro (UDC, Centre Union), Popolo delle Libertà (PDL, People for Freedom), Lega Nord (LN, Northern League). The ordering of the parties is assigned according to their willingness to form a coalition in the recent political history.

Table 2. Seats allocation in the Camera (April 2008) for the five main parties

Parties	IdV	PD	UDC	PDL	LN
Seats	28	218	34	272	60

Supposing that all the Deputies of the five main parties vote, so the majority quota is 307, we may represent the Camera as the weighted majority situation [307; 28, 218, 34, 272, 60]. In order to compute the Shapley-Shubik index using the relations in (12) and (13), we need the data in Table 3 (for each coalition the critical parties are underlined and $\beta_{ij} = 1/c_i$ for each critical party j in each coalition $S_i \in W$).

Looking at the parameters α_i, we remark how the actual majority coalition $\{4,5\}$ has not the highest probability to form, while the most probable coalition is $\{1,2,3,5\}$, that includes the leftmost and rightmost parties (IdV and LN) and excludes the relative majority party (PDL).

The Shapley-Shubik index is

$$\phi(w) = \left(\frac{2}{60}, \frac{12}{60}, \frac{7}{60}, \frac{27}{60}, \frac{12}{60}\right).$$

Using the procedure previously described, we modify the parameters α_i according

Table 3. Parameters assigned to the winning coalitions to compute the Shapley-Shubik index

S_i	$\underline{24}$	$\underline{45}$	$1\underline{24}$	$\underline{134}$	$1\underline{45}$	$\underline{2}3\underline{4}$	$\underline{235}$	$2\underline{4}5$	$3\underline{45}$	$123\underline{4}$	$1\underline{235}$	$12\underline{4}5$	$13\underline{4}5$	2345	12345
$p(S_i)$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{5}$
α_i	$\frac{6}{60}$	$\frac{6}{60}$	$\frac{4}{60}$	$\frac{6}{60}$	$\frac{4}{60}$	$\frac{4}{60}$	$\frac{6}{60}$	$\frac{2}{60}$	$\frac{4}{60}$	$\frac{3}{60}$	$\frac{9}{60}$	$\frac{3}{60}$	$\frac{3}{60}$	0	0
β_{ij}	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	1	$\frac{1}{2}$	1	$\frac{1}{3}$	1	1		

to (16) and compute the relative FP index given by (15)

$$\phi^{FP}(w) = \left(0, \frac{2}{17}, 0, \frac{10}{17}, \frac{5}{17}\right).$$

Figure 1 shows the first steps of the procedure of reducing the probability to form of the non-contiguous coalitions.

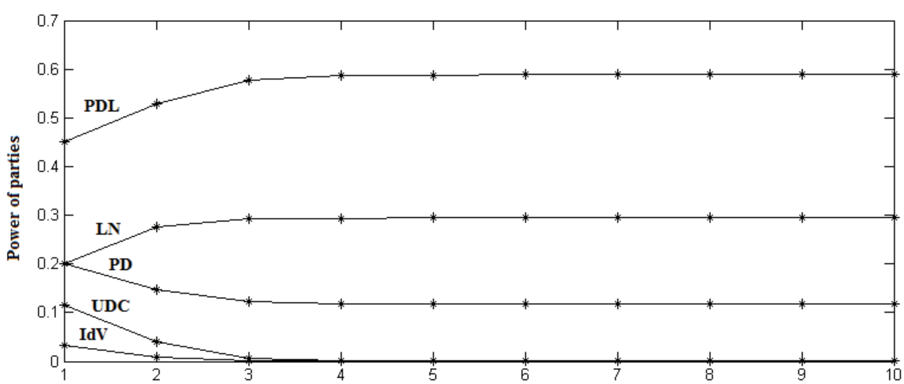

Figure 1. First 10 steps of the procedure for reducing the probability to form of the non-contiguous coalitions for the Shapley-Shubik index

We complete the example computing the power given to the five parties by the other classical indices and the modified power obtained with the *FP* version. We summarize the results in Table 4.

The Shapley-Shubik and the Banzhaf-Coleman indices assign a positive power to the small parties, IdV and UDC, as they are critical for some winning coalitions ($\{1,3,4\}$, $\{2,3,5\}$ and $\{1,2,3,5\}$), none of which is contiguous. Consequently, the power of these two parties, with the modified indices which take into account only when a player is critical for a contiguous winning coalition, goes down to zero, even if one of them, UDC, has an intermediate position on the left-right axis. Allowing only connected coalitions to form, we expect that this should be beneficial for the centrist

Table 4. Main power indices in the classical and in the modified version

Parties	IdV	PD	UDC	PDL	LN
$\phi(w)$	$\frac{2}{60}$	$\frac{12}{60}$	$\frac{7}{60}$	$\frac{27}{60}$	$\frac{12}{60}$
$\phi^{FP}(w)$	0	$\frac{2}{17}$	0	$\frac{10}{17}$	$\frac{5}{17}$
$\beta(w)$	$\frac{1}{25}$	$\frac{5}{25}$	$\frac{3}{25}$	$\frac{11}{25}$	$\frac{5}{25}$
$\beta^{FP}(w)$	0	$\frac{1}{7}$	0	$\frac{4}{7}$	$\frac{2}{7}$
$\delta(w)$	$\frac{2}{24}$	$\frac{5}{24}$	$\frac{4}{24}$	$\frac{8}{24}$	$\frac{5}{24}$
$\delta^{FP}(w)$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
$H(w)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{2}{10}$
$H^{FP}(w)$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$

parties, so it is not surprising to assign zero to IdV, as it does not have a central position in the Parliament, but it seems strange, at a first look, the zero given to UDC. The reason why of this result is given by the fact that the dummy property filters through contiguity: a dummy player will remain so, but it is possible that a non dummy player, as UDC, which has non zero marginal contribution only in non-contiguous coalitions, becomes dummy once we evaluate the embedded Shapley-Shubik and Banzhaf-Coleman indices. This is not true anymore for a generic *FP* index, which does not account the philosophy of the marginal contribution. If we allow a player to be critical not only when it makes a winning coalition being losing by leaving, but also when it makes a contiguous winning coalition being non-contiguous by leaving, it is possible that such an index "creates" power (i.e. it assigns a positive power to a dummy player).

It is interesting to observe how the indices $\phi^{FP}(w)$ and $\beta^{FP}(w)$ give a higher power to the two parties of the actual majority coalition, PDL and LN (greater for PDL, the party with the relative majority quota of seats), guaranteeing a positive power to the second party of the Camera for number of seats, PD, which is critical for the contiguous coalition $\{2,3,4\}$.

Once we take into account only minimal winning coalitions, as we do when we evaluate the Deegan-Packel and the Holler indices, we notice that every party belongs to at least one of them. But focusing on the contiguous ones, the unique contiguous minimal winning coalition is the actual majority PDL and LN and the power is equally shared between these two parties, as they are in a symmetric position.

7. Cooperation structures

In this section, we relax the hypothesis of contiguity of the parties. We represent the new situation using a graph similar to the cooperation structure introduced by Myerson (1977).

We consider a non-oriented graph whose vertices are the parties and whose edges represent the willingness of the parties corresponding to the vertices to reach an agreement taking into account their ideological positions. We denote an edge between parties k and h by $k:h$. Let $g^N = \{k:h | k \in N, h \in N, k \neq h\}$ be the complete graph and let $G^N = \{g | g \subseteq g^N\}$ be the set of all graphs on N, each one representing a possible political situation involving the parties at hand.

Given a subset of parties $S \subseteq N$ and a graph $g \in G^N$, we say that $k,h \in S$ are *connected in S by g* if there exists a path in g from k to h, i.e. a sequence $(k^0, \ldots, k^i)$ such that $k^0 = k$, $k^i = h$ and $k^{j-1} : k^j \in g$ and $k^j \in S$ for $j = 1, \ldots, i$.

A coalition $S \subseteq N$ is *connected by g* if all pairs $k,h \in S$ are connected in S by g. The central role played by the contiguous coalitions is now assigned to the connected ones, so we assume that W^c represents the set of winning coalitions that are connected by the graph g. Our model of the left-right axis is a particular case of coalition structure given by a line-graph. The corresponding graph g' is shown in Figure 2.

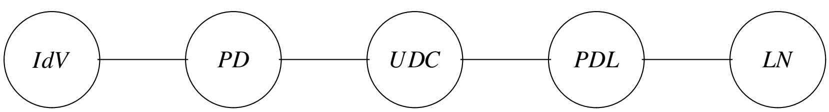

Figure 2. Line-graph g' of the left-right axis situation

Given a graph $g \in G^N$ we extend the *FP* family defining a new family, denoted by $\widetilde{FP}$, based on the set of the coalitions connected by g, their relative probability to form and the power share inside each coalition.

We may apply the modified procedure to compute the power assigned to the parties of the example in Section 6, after adding the edge PD:LN. We remark that in the actual Italian political situation these two parties have far ideologies, but such an agreement took place in 1996. This cooperation structure is represented by Figure 3, denoted by g''.

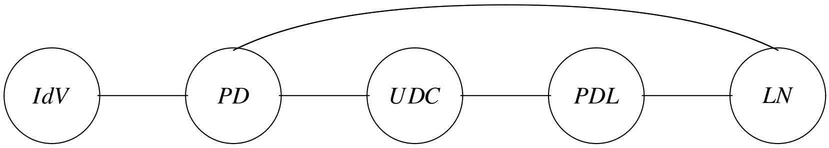

Figure 3. Graph g'' of the modified situation

The computation of the new limit values is given in Table 5.

Table 5. Modified indices related to the connection structure given by graph g''

Parties	IdV	PD	UDC	PDL	LN
$\phi^{\widetilde{FP}}$	0	$\frac{7}{40}$	$\frac{5}{40}$	$\frac{18}{40}$	$\frac{10}{40}$
$\beta^{\widetilde{FP}}$	0	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{7}{16}$	$\frac{4}{16}$
$\delta^{\widetilde{FP}}$	0	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{5}{12}$
$H^{\widetilde{FP}}$	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$

Comparing the results in Tables 4 and 5, we may notice that all the indices related to g'' reduce the power of the main party, PDL, after the introduction of the new edge. In the new situation also coalitions that do not include PDL may form.

For a complete analysis of the results we obtained, as we have largely used the idea of a cooperation structure by Myerson, we compute the Myerson index (see Myerson 1977) for both the situations g' and g'' in order to underline some important differences between the two ideas of solution. The results are shown in Table 6.

Table 6. The Myerson indices for the game (N, w) of the example of Section 6 referred to the graph g' and g''

Parties	IdV	PD	UDC	PDL	LN
$M(w, g')$	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{7}{12}$	$\frac{3}{12}$
$M(w, g'')$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$

As expected, the *FP* index and the Myerson index give different vectors of power. In particular, we can underline how in our model not necessarily both the parties, PD and LN, should have the same advantage by allowing a possible agreement between them. We notice that in this case PD has a higher power when the cooperation structure is given by g'', while LN should prefer the g' situation. This could not have been happened using Myerson index as the property of *equity*[1] is always guaranteed. The lost of power of LN shows that even the property of *total stability*[2] no longer holds.

[1] According to equity, introducing (or removing) an edge, both the parties corresponding to its extreme vertices have the same variation of power.

[2] According to stability, after introducing an edge, the variation of power is non negative for both its extreme vertices.

8. Concluding remarks

In Section 4 we recalled the definition of the *FP* family of power indices. As we showed in Section 7, the embedded Shapley-Shubik index differs from the idea of Myerson, in which non-connected coalitions are feasible. On the opposite we consider that these coalitions cannot form. Also the idea of *a priori* unions given by Owen, which takes into account a restricted set of orderings, differs from our model. As it has already been shown in Fragnelli et al. (2009), the way this family approaches the problem is quite far from the ones of the classical indices. By an extension of the *FP* family into a new family $\overline{FP}$ and by the definition of a sequence of power indices assigning a reduced probability to form to the non-contiguous coalitions, it was possible to profit both of some properties of the classical indices and of the new idea of accounting the relevance of the contiguous coalitions. In particular we may observe the loss of monotonicity during the embedding procedure: the issue of contiguity is "versus" monotonicity, as the connection degree may be more relevant that the number of seats. The non-monotonicity can be simply seen in the example in Figure 4.

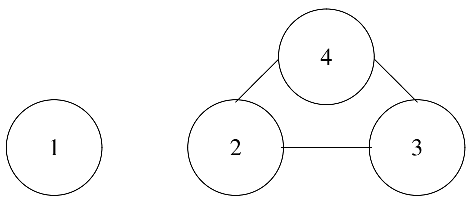

Figure 4. Graph of a non-monotonic situation

We assume that the game is described by weighted majority situation $[6;4,2,2,2]$. In this situation, party 1, the one with the highest weight, has zero power with every FP index, as it does not belong to any contiguous winning coalition.

Then, in Section 7, we went further, considering coalitions connected according to a given graph. This enables us to account for multidimensional situations, instead of the simple left-right axis.

The idea of giving zero probability to form to the non-contiguous coalitions can be a strong assumption. It can be observed that, even if it is not very common that parties with quite different political ideologies can decide to cooperate, it is still possible they have the necessity to negotiate and make an agreement in some particular situations. The procedure we showed to obtain an FP index, starting from an $\overline{FP}$ one, is based on the idea of putting down to zero the probability of the non-contiguous coalitions to form. Using a sequence of vectors, $(\overline{FP})_t$, that for each value of $t \in \mathbb{N}_>$ provides a power index, where the non-contiguous coalitions have a reduced, but positive, probability.

It should be clear that the main point is to have a sequence of probability distributions that reduces to zero the probability of non-contiguous coalitions, leaving a positive probability to some contiguous coalitions. Consequently, we could make use of any sequence that satisfies these requirements. The one we proposed in Section 5 is only a simple way to accomplish the requests. For instance, it is possible to use different criteria of convergence to zero of non-contiguous coalitions, taking into account the ideological distance among the parties and to assign different probabilities to form to the contiguous ones. In particular, via a suitable analysis of real data, we can choose any vector $(\overline{FP})_t$ selecting an appropriate value for t. Of course, the approach of a sequence of values may be replaced by defining a distribution of probabilities that directly assigns zero to non-contiguous coalitions. This idea is very simple but does not provide us a sequence of power indices. Moreover, the probabilities to form of contiguous coalitions may result in an index that no longer embeds the original one. Finally, a different sequence of probability distributions may give rise to questions about the sensitivity analysis of the resulting indices.

The comparison with the Myerson value suggested us the possibility to obtain the embedded power indices by defining a new game in which the characteristic function w is modified as w' s.t. $w'(S) = w(S)$ if $S \in W^c$ and $w'(S) = 0$ otherwise and evaluating the classical power indices on this new game. We can immediately notice that (N, w') is not a simple game anymore, as it loses the property of monotonicity. Also evaluating the Shapley value (Shapley 1953), which can be applied to every game, we may obtain negative values for some parties, so that it may hardly be considered as a measure of relevance.

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