Cooperation in stochastic inventory models with continuous review

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Abstract

Consider multiple companies that continuously review their inventories and face Poisson demand. We study cooperation strategies for these companies and analyse if there exist allocations of the joint cost such that any company has lower costs than on its own; such allocations are called stable cost allocations. We start with two companies that jointly place an order for replenishment if their joint inventory position reaches a certain reorder level. This strategy leads to a simple expression of the joint costs. However, these costs exceed the costs for non-cooperating companies. Therefore, we examine another cooperation strategy. Namely, the companies reorder as soon as one of them reaches its reorder level. This latter strategy has lower costs than for non-cooperating companies. Numerical experiments show that the game-theoretical distribution rule — a cost allocation in which the companies share the procurement cost and each pays its own holding cost — is a stable cost allocation. These results also hold for situations with multiple companies.

Key words: joint replenishment, stochastic demand, cost allocation, distribution rule, continuous review, game theory, inventory model.

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1 Introduction

Several companies may have the same item on stock to meet the demands of their customers. Instead of working on their own, the companies may jointly place an order for replenishment of their stocks and save on procurement costs. The main questions in such joint replenishment problems are how much to order, when to order and how to allocate the cost savings.

This paper starts with the analysis of situations with two companies. The companies jointly place an order for replenishment if their joint inventory position reaches a certain reorder level. This strategy leads to a simple expression of the joint cost. However, this cost exceeds the total costs for non-cooperating companies. Hence, for fruitful cooperation a more complex strategy is needed. We continue our analysis with another cooperation strategy: the companies reorder as soon as one of them reaches its reorder level. This strategy has lower costs than for non-cooperating companies. Hence, the companies are willing to cooperate since this strategy reduces the total cost. A natural question that arises is how to allocate the joint costs of cooperation among the companies.

For cost allocations, a first requirement is that they are *stable*. Stability means that any group of companies pays at most its inventory cost; the group has no incentives to disagree with the cost allocation. This requirement is very well represented by the core — a stability concept from cooperative game theory. The core is a set of cost allocations such that no group of companies pays more in this allocation than what they pay on their own. Often, there are many stable allocations. Then the companies should select one of these. Two specific cost allocations are the Shapley value [13], and the distribution rule, which is inspired by the same rule for deterministic inventory situations [12]. The Shapley value is a cost allocation that distributes marginal contributions to the costs equally among the companies. The distribution rule consists of two parts. Namely, (i) the joint procurement costs are allocated in a proportional way to the companies, and (ii) each company pays its individual inventory holding costs under cooperation. Our numerical experiments show that both these cost allocations are stable.

There is a large literature on joint replenishment problems. The papers [2, 3, 7] are excellent surveys on this subject. One of the early papers on joint replenishment is [15], in which a situation is studied where two items are stocked. These items have *identical* demand and cost characteristics. The lead time is assumed to be zero. [10] was inspired by [15]. There also optimal stationary ordering policies are studied for a two product inventory system with continuous review. Special interest goes to random joint policies (S, c, s): whenever the inventory of any product drops to its reorder point s, all products with inventory less than or equal to their can-order points c are ordered up to their order-to point S. [6] extends this analysis to companies facing compound stochastic demand. More recent, [16] and [5] study classes of replenishment policies that lead to low joint costs compared to other well-known policies. None of these papers considers allocations of the joint costs.

There are very few papers that analyse the stability of cost allocations for joint replenishment problems by means of cooperative game theory. One of these papers is [9] that studies allocations of joint costs in inventory models under continuous review. Their model is different from ours, since they assume that the companies set up a central warehouse to store their goods and to meet the demand of their customers. In the case of three companies they show that there exists a cost allocation that is stable, justifiable (the allocated cost is in line with the cost savings) and computable in polynomial time. In [12] cooperation in an inventory system with stationary deterministic demand is studied. The companies face procurement and holding costs. Further, the authors introduce the distribution rule, which is a cost allocation designed for joint replenishment problems. The authors show that this cost allocation is stable. In the present paper, we show that this result also holds for inventory systems with stochastic (Poisson) demand and integer order quantities. Also related is the thesis [4], which contains an initial study on our model with some other cost allocations. A review on general game theoretical applications in supply chain management may be found in [11].

The outline of this paper is as follows. In section 2 we introduce our model. Section 3 analyses the costs of non-cooperating companies. In section 4 a first, simple, cooperation strategy is analysed for two companies. Namely, the companies place a joint order for replenishment of their stocks if their joint inventory position reaches a certain level. Although this is an easy strategy, it leads to large costs. Therefore, in section 5 another cooperation strategy for two companies is analysed. Here, the companies reorder if one of them reaches its reorder level. This strategy has lower costs than for non-cooperating companies. In case of two identical companies we characterize the values of the procurement cost such that cooperation is beneficial. Section 6 studies cost allocations. Numerical studies show that the Shapley value and the distribution rule are stable cost allocations. In section 7 we analyse the second cooperation strategy for multiple companies. Numerical results show that also for three companies both the Shapley value and the distribution rule are stable cost allocations. Section 8 concludes. An appendix contains the proofs that are omitted from the text.

2 Model

We consider the inventory control problem model of a single product for multiple companies under continuous review. Let N denote the finite set of companies. The demand for the product at the different companies occurs in discrete units, and the demand processes are independent Poisson processes with rate λ_i for company i, $i \in N$.

To meet their demands the companies place orders for replenishment of their stocks. We assume that the lead time of an order is zero time units and that backorders are not allowed. The replenishment policy for company i is to place an order for Q_i items when its inventory position falls below r_i items. Hence, such a policy is defined by the reorder level r_i and the order quantity Q_i. The state of company i is determined by the inventory position Z_i^t at time t. Let $\mathbb{E}Z_i^{non-coop}$ denote the expectation of Z_i^t in steady state, and let Z_i denote the steady state random variable.

To evaluate the benefits of cooperation, let us specify the cost structure of the companies. We identify procurement costs and inventory holding costs. First, the procurement costs are the costs associated with procuring (replenishing) the units stocked. We assume that each replenishment order (either by a single company or by multiple cooperating companies) incurs the fixed procurement cost A.

Second, the inventory holding costs are the costs of carrying the items in inventory. Let company i have holding cost h_i per unit in stock per time unit. Then the inventory holding costs equal $h_i Z_i^t$ per time unit when the inventory at time t equals Z_i^t. Because the lead time is zero, companies use $r_i = 0$ — an order for replenishment is placed as soon as the company runs out of inventory — to minimize costs.

3 Non-cooperating companies

In this section we consider non-cooperating companies that place their orders independently. Company i uses the following replenishment policy: place an order for an amount Q_i each time the inventory position reaches level 0. It is obvious that the inventory position processes $\{Z_i^t\}_{i \in N}$ of the companies are independent processes, and that process Z_i^t has state space $S_i = \{n_i : 1 \leq n_i \leq Q_i\}$. For completeness, and to support the more complicated expressions for cooperating companies, we review below the results for a single company.

The marginal inventory position equilibrium distribution is readily seen to be (see e.g.

[8, p. 183])

$$v_i(j) = \lim_{t\to\infty} P(Z_i^t = j) = \frac{1}{Q_i}, \quad j = 1, ..., Q_i,$$

with expectation $\mathbb{E}Z_i^{non-coop} = \frac{1}{2}(Q_i + 1)$.

The expected procurement cost per unit time can readily be obtained from a renewal argument. The inventory position process Z_i^t forms a renewal process that regenerates each time an order is placed. Thus, the long run average procurement cost is

$$\lim_{t\to\infty} \frac{1}{t} AN_i(t) = A\frac{\lambda_i}{Q_i},$$

where $N_i(t)$ is the number of replenishment orders in the time interval $(0, t]$. By the renewal property $\lim_{t\to\infty} N_i(t)/t = 1/\mathbb{E}T_i$ with probability 1, where T_i is the cycle time for company i. It is readily seen that $\mathbb{E}T_i = Q_i/\lambda_i$.

The total expected cost rate $K_i(Q_i)$ for company i is

$$K_i(Q_i) = A\frac{\lambda_i}{Q_i} + h_i\mathbb{E}Z_i^{non-coop} = \frac{A\lambda_i}{Q_i} + \frac{1}{2}h_i(Q_i + 1). \tag{1}$$

This is a convex function in Q_i. Company i will select an integer quantity that minimizes this expected cost per time unit. This quantity Q_i^{nc} is called the optimal replenishment quantity for company i, and it equals $\lfloor x \rfloor$ or $\lceil x \rceil$ with $x = \sqrt{2A\lambda_i/h_i}$.

4 Joint replenishment under sum constraint

In this section we consider two cooperating companies. Cooperation means that the companies join their orders for replenishment of their inventories; this way they save on procurement costs. We may invoke several cooperation strategies. First assume that both companies jointly place an order as soon as their joint inventory position falls below a certain level. We call this cooperation under the sum constraint. A second cooperation strategy is studied in Section 5.

The model for two cooperating companies is as follows. The pair of inventory positions of the companies immediately after the arrival of an order equals (Q_1, Q_2). Demand for both firms occurs until the combined inventory position $Z_1^t + Z_2^t$ falls below a joint reorder level. Demand for company 2 only implies that a new order for replenishment may be placed when a new demand arrives at $(Q_1, 1)$, after Q_2 demands for company 2, because $r_i = 0$. Demand for company 1 only implies that an order may be placed when a new demand for company 1 arrives at $(1, Q_2)$, after Q_1 demands for company 1. To ensure nonnegative inventory positions, the joint inventory level should stay on or above $\max\{Q_1+1, 1+Q_2\}$; this value is defined to be the reorder level. Notice that a larger reorder level leads to larger holding costs, and is thus not desirable. Rewritten, the joint reorder level equals

$$\max\{Q_1, Q_2\} + 1.$$

Without loss of generality, assume that $Q_1 \geq Q_2$. Then the state space $\tilde{S}$ is upper triangular,

$$\tilde{S} = \{(n_1, n_2) : n_i \leq Q_i, i = 1, 2;\ n_1 + n_2 \geq Q_1 + 1\},$$

where n_i denotes the inventory position of company i. A joint order for replenishment of stock is placed if the joint inventory position $n_1 + n_2$ drops to Q_1. Thus, the joint cumulative demand during a cycle equals Q_2.

The joint inventory position (n_1, n_2) evolves as a continuous time Markov chain at the state space $\tilde{S}$. Let $\tilde{\pi}(n_1, n_2)$ denote the steady state probability for state (n_1, n_2). There are three types of states in $\tilde{S}$ namely inner states, boundary states and the regeneration state. Inner states are states in $\tilde{S}_I = \left\{(n_1, n_2) \in \tilde{S} : 1 \leq n_i < Q_i\right\}$ where for both companies demand has occurred. The flow balance equations for these states ("rate out equals rate in") are

$$(\lambda_1 + \lambda_2)\tilde{\pi}(n_1, n_2) = \lambda_1 \tilde{\pi}(n_1 + 1, n_2) + \lambda_2 \tilde{\pi}(n_1, n_2 + 1) \tag{2}$$

for all $(n_1, n_2) \in \tilde{S}_I$. Second, $\tilde{S}_B = \left\{(n_1, n_2) \in \tilde{S} : n_i = Q_i,\ n_j < Q_j,\ i \neq j\right\}$ is the set of boundary states in which demand has occurred for only one firm. The flow balance equations for these states are

$$(\lambda_1 + \lambda_2)\tilde{\pi}(n_1, Q_2) = \lambda_1 \tilde{\pi}(n_1 + 1, Q_2) \tag{3}$$

if $n_1 < Q_1$ $(j = 1)$ and

$$(\lambda_1 + \lambda_2)\tilde{\pi}(Q_1, n_2) = \lambda_2 \tilde{\pi}(Q_1, n_2 + 1) \tag{4}$$

if $n_2 < Q_2$ $(j = 2)$. The final type of state is the regeneration point $\tilde{S}_R = \{(Q_1, Q_2)\}$ that is reached each time a joint order arrives. For this state the flows satisfy

$$(\lambda_1 + \lambda_2)\tilde{\pi}(Q_1, Q_2) = \lambda_1 \sum_{n_2=1}^{Q_2} \tilde{\pi}(1, n_2) + \lambda_2 \sum_{n_1=1}^{Q_1} \tilde{\pi}(n_1, 1). \tag{5}$$

These balance equations (2)-(5) determine the equilibrium distribution.

Lemma 1 *The equilibrium distribution $\tilde{\pi}$ is a truncated binomial distribution:*

$$\tilde{\pi}(n_1, n_2) = Q_2^{-1}\rho(n_1, n_2)$$

for all $(n_1, n_2) \in \tilde{S}$, with

$$\rho(n_1, n_2) = \begin{pmatrix} Q_1 - n_1 + Q_2 - n_2 \\ Q_1 - n_1 \end{pmatrix} p^{Q_1 - n_1}(1-p)^{Q_2 - n_2}, \tag{6}$$

and $p = \lambda_1/(\lambda_1 + \lambda_2)$ is the proportion of demand for company 1.

The proof of this lemma is in appendix A.

We can now determine the expected length of a cycle.

Lemma 2 *The expected cycle time is $Q_2/(\lambda_1 + \lambda_2)$.*

Proof. Let T be the joint cycle time under cooperation. Then $P(T > t)$ is the probability that the cycle did not end by time t. Hence, the joint demand in the cycle up to time t is smaller than Q_2. Since the joint demand is Poisson distributed with rate $\lambda_1 + \lambda_2$ we derive

$$\begin{aligned} \mathbb{E}T &= \int_0^\infty P(T > t)dt = \int_0^\infty \sum_{k=0}^{Q_2-1} \frac{((\lambda_1+\lambda_2)t)^k}{k!} e^{-(\lambda_1+\lambda_2)t} dt \\ &= \frac{1}{(\lambda_1+\lambda_2)} \sum_{k=0}^{Q_2-1} \int_0^\infty \frac{u^k}{k!} e^{-u} du = \frac{Q_2}{(\lambda_1+\lambda_2)}. \end{aligned}$$

□

The expression for the expected cycle time is natural since it is the expected time until replenishment, that is, until Q_2 demands have occurred.

The lemma below shows the joint cost rate for the companies if they use the cooperation strategy under the sum constraint. The proof of this lemma is in appendix A.

Lemma 3 *Consider the cooperation strategy under the sum constraint. The expected joint cost per time unit given order quantities (Q_1, Q_2) equals*

$$\tilde{K}(Q_1, Q_2) = A\frac{\lambda_1 + \lambda_2}{Q_2} + \frac{1}{2}h_1\left(2Q_1 - p(Q_2 - 1)\right) + \frac{1}{2}h_2\left(Q_2 + 1 + p(Q_2 - 1)\right).$$

Note the similarity with the cost for non-cooperating companies in (1). Minimizing this expected cost results in equal order quantities for both companies.

Corollary 4 *If the quantities* (Q_1^s, Q_2^s) *minimize the cost* $\tilde{K}(Q_1, Q_2)$ *then* $Q_1^s = Q_2^s$*, and this optimal quantity is* $\lfloor x \rfloor$ *or* $\lceil x \rceil$ *with* $x = \sqrt{2A(\lambda_1 + \lambda_2)/(h_1(2-p) + h_2(1+p))}$*.*

Proof. The statement $Q_1^s = Q_2^s$ follows directly from lemma 3 and $Q_1 \geq Q_2$. Let $Q^s = Q_i^s$. Then

$$\tilde{K}(Q^s, Q^s) = A\frac{\lambda_1 + \lambda_2}{Q^s} + \frac{1}{2}h_1\left((2-p)Q^s + p\right) + \frac{1}{2}h_2\left((1+p)Q^s + 1 - p\right).$$

This is a convex function with real optimal quantity $\sqrt{2A(\lambda_1 + \lambda_2)/(h_1(2-p) + h_2(1+p))}$. The integer optimum is one of the neighboring integers. □

We proceed by analysing whether it is worthwhile to use this strategy. For this, we compare the joint cost under cooperation with the sum constraint with the cost for non-cooperating companies.

Theorem 5 *The total optimal cost for non-cooperating companies is lower than for cooperation under the sum constraint. Namely* $K_1(Q_1^{nc}) + K_2(Q_2^{nc}) < \tilde{K}(Q^s, Q^s)$ *if* $Q^s > 1$*, and* $K_1(Q_1^{nc}) + K_2(Q_2^{nc}) \leq \tilde{K}(Q^s, Q^s)$ *if* $Q^s = 1$*.*

The proof of this theorem is in appendix A.

Cooperation under the sum-constraint is a strategy that leads to a simple expression for the joint cost, as shown in lemma 3. However, as theorem 5 above shows, this cooperation strategy yields larger costs than for non-cooperating companies; it is not worthwhile to use this strategy. Therefore, in the next section we introduce another cooperation strategy that does lead to lower costs compared to individual optimization.

5 Joint replenishment under individual constraints

In this section an alternative replenishment strategy is studied. In section 5.1 this strategy is studied for two companies in general, and in section 5.2 for two identical companies.

5.1 General model for two companies

An alternative joint replenishment strategy is that both companies reorder as soon as one of the companies reaches its individual reorder level. Now the state space is the set

$$S = \{(n_1, n_2) : 1 \leq n_i \leq Q_i,\ i = 1, 2\}.$$

A joint order for replenishment is placed as soon as the inventory position of a company equals zero. That is, the cumulative demand for this company is Q_i units.

Lemma 6 *The equilibrium distribution π is a truncated binomial distribution:*

$$\begin{aligned}\pi(n_1,n_2) &= \lim_{t\to\infty} P\big((Z_1^t, Z_2^t) = (n_1,n_2)\big) \\ &= \frac{1}{G(Q_1,Q_2)} \left(\begin{array}{c} Q_1 - n_1 + Q_2 - n_2 \\ Q_1 - n_1 \end{array} \right) p^{Q_1-n_1}(1-p)^{Q_2-n_2},\end{aligned}$$

for states $(n_1,n_2) \in S$ where

$$G(Q_1,Q_2) = \sum_{z_1=0}^{Q_1-1} \sum_{z_2=0}^{Q_2-1} \left(\begin{array}{c} z_1 + z_2 \\ z_1 \end{array} \right) p^{z_1}(1-p)^{z_2}$$

is the normalising constant.

Proof. The equations (2)-(5) are the global balance equations if we replace the state space $\tilde{S}$ by S. Substitution of the equilibrium distribution into these equations yields the result immediately. The expression for the normalising constant uses the substitution $z_i = Q_i - n_i$. $\square$

Let us now consider the cycle time for the system with two companies. Let T be the random variable denoting the time between two replenishments in the system. For the cycle time we obtain a result that closely resembles that for non-cooperating companies.

Lemma 7 *The expected cycle time is $\mathbb{E}T = G(Q_1,Q_2)/(\lambda_1+\lambda_2)$.*

Proof. Let T_i be the cycle time for company i as if it would operate on its own. Then T_1 and T_2 are independent variables denoting the time from a replenishment until the process reaches a reorder level of one of the companies. In particular, if $T_i > t$ then company i did not reach its reorder level yet, so, the total demand for company i so far during this cycle is below Q_i. We can now compute the expected cycle time.

$$\begin{aligned}\mathbb{E}T &= \int_0^\infty P(T>t)dt \;=\; \int_0^\infty P(T_1>t)P(T_2>t)dt \\ &= \int_0^\infty \sum_{k=0}^{Q_1-1} \frac{(\lambda_1 t)^k}{k!} e^{-\lambda_1 t} \sum_{\ell=0}^{Q_2-1} \frac{(\lambda_2 t)^\ell}{\ell!} e^{-\lambda_2 t} dt \\ &= \frac{1}{\lambda_1+\lambda_2} \sum_{k=0}^{Q_1-1} \sum_{\ell=0}^{Q_2-1} \binom{k+\ell}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^\ell \int_0^\infty \frac{t^{k+\ell}}{(k+\ell)!} e^{-t} dt \\ &= \frac{1}{\lambda_1+\lambda_2} \sum_{k=0}^{Q_1-1} \sum_{\ell=0}^{Q_2-1} \binom{k+\ell}{k} p^k (1-p)^\ell \;=\; \frac{G(Q_1,Q_2)}{\lambda_1+\lambda_2}\end{aligned}$$

□

Define $P(n_1, n_2)$ as the probability that the system reaches the state (n_1, n_2) in a cycle. Clearly, the regeneration point is reached for sure — $P(Q_1, Q_2) = 1$ — and

$$P(n_1, n_2) = pP(n_1 + 1, n_2) + (1 - p)P(n_1, n_2 + 1).$$

Hence, recall (6),

$$P(n_1, n_2) = \rho(n_1, n_2) = \binom{Q_1 - n_1 + Q_2 - n_2}{Q_1 - n_1} p^{Q_1 - n_1}(1 - p)^{Q_2 - n_2}$$

for $(n_1, n_2) \in S$.

Further, define the probability $P(0, n_2)$ as the probability that the cycle terminates from state $(1, n_2)$ due to an arrival of a demand for company 1 completing the cycle; $P(0, n_2) = pP(1, n_2)$. Similarly, define $P(n_1, 0) = (1 - p)P(n_1, 1)$. The probability that firm 1 ends a cycle is

$$\sum_{n_2=1}^{Q_2} P(0, n_2) = \sum_{z_2=0}^{Q_2-1} \binom{Q_1 - 1 + z_2}{z_2} p^{Q_1}(1 - p)^{z_2} = I_p(Q_1, Q_2),$$

where

$$I_q(a, b) = \sum_{s=0}^{b-1} \binom{s + a - 1}{s} q^a (1 - q)^s \tag{7}$$

is the generalized incomplete beta function [1, section 26.5], [17]. Further,

$$\sum_{n_1=1}^{Q_1} P(n_1, 0) = \sum_{z_1=0}^{Q_1-1} \binom{z_1 + Q_2 - 1}{z_1} p^{z_1}(1 - p)^{Q_2} = I_{1-p}(Q_2, Q_1)$$

is the probability that firm 2 ends the cycle. The cycle terminates via a demand for company 1 or 2 and therefore these probabilities sum to 1; this follows directly from the property

$$I_q(a, b) + I_{1-q}(b, a) = 1 \tag{8}$$

of the generalized incomplete beta function.

These probabilities allow for an alternative formulation of the expected cycle time and the normalizing constant of the equilibrium distribution.

Lemma 8 *The expected cycle time may be written as*

$$\mathbb{E}T = \frac{Q_2}{\lambda_2} I_{1-p}(Q_2+1, Q_1) + \frac{Q_1}{\lambda_1} I_p(Q_1+1, Q_2),$$

and

$$G(Q_1, Q_2) = \frac{Q_2}{1-p} I_{1-p}(Q_2+1, Q_1) + \frac{Q_1}{p} I_p(Q_1+1, Q_2)$$

is an alternative expression for the normalising constant.

The proof of this lemma is in appendix A.

The joint cost for the companies per cycle are as follows.

Lemma 9 *In case of cooperation under individual constraints, the expected joint cost per time unit given order quantities* (Q_1, Q_2) *equals*

$$\begin{aligned} K(Q_1, Q_2) &= A(\lambda_1 + \lambda_2)/G(Q_1, Q_2) \\ &+ \frac{Q_2}{G(Q_1, Q_2)} \sum_{z_1=0}^{Q_1-1} [h_1(Q_1 - z_1/2) + h_2(Q_2+1)/2] \binom{z_1 + Q_2}{z_1} p^{z_1} (1-p)^{Q_2} \\ &+ \frac{Q_1}{G(Q_1, Q_2)} \sum_{z_2=0}^{Q_2-1} [h_1(Q_1+1)/2 + h_2(Q_2 - z_2/2)] \binom{Q_1 + z_2}{z_2} p^{Q_1} (1-p)^{z_2}. \end{aligned}$$

The proof of this lemma is in appendix A.

The companies minimize the costs K by selecting a suitable pair (Q_1, Q_2) of integer order quantities. This leads to lower costs than in case of cooperation under the sum constraints.

Theorem 10 *Cooperation under the individual constraints leads to lower expected joint costs than cooperation under the sum constraints.*

Proof. Consider a joint inventory situation with parameters A, λ_1, λ_2, h_1 and h_2. Let (Q^s, Q^s) be the optimal strategy under cooperation with the sum-constraint. We show that cooperation under the individual constraints leads to lower costs compared to cooperation under the sum constraint.

First, when cooperating under the individual constraints the joint inventory position ranges from $2Q^s$ down to $n_i + 1$ with n_i between 1 and Q^s, for some company i. When the companies cooperate under the sum constraint then the joint inventory position ranges from $2Q^s$ down to $Q^s + 1$. This lower bound is larger than for cooperation under the individual constraints. Therefore, the average inventory position of both companies are larger than for cooperation under the sum constraints.

Second, in case of individual constraints inventory is replenished when the inventory position of one of the firms — say firm i — drops to 0; the joint accumulated demand equals $2Q^s - n_j$, $j \neq i$. In case of the sum constraint, replenishment occurs when the joint inventory position reaches Q^s. Then, the joint accumulated demand equals Q^s. This is lower than in case of individual constraints, $Q^s \leq 2Q^s - n_j$. Therefore, the cycle time is lower than for cooperation under the individual constraints. Together with the first result this implies that both the holding cost per time unit and the order cost per time unit are lower under cooperation with individual constraints. □

The next step is to compare cooperation under the individual constraints with individual optimization. Cooperation does not increase the optimal order quantity.

Theorem 11 *For any company i, the optimal order quantity under cooperation with the individual constraints does not exceed the individual optimal order quantity Q_i^{nc}.*

Proof. Consider $i = 1$. Assume that the order quantity of company 2, Q_2, is fixed. Then under cooperation there is a positive probability that firm 2 ends the cycle and initiates a new joint order. In that case, the cycle ends before company 1 has reached its reorder level; its inventory position is rather high, leading to rather large holding costs. A lower order quantity would decrease these costs. If under cooperation firm 1 ends the cycle, then the situation is the same as under individual optimization. Hence, the optimum order quantity for company 1 under cooperation does not exceed the individual optimal quantity Q_1^{nc}. □

It is difficult to analyse the difference between the joint cost of cooperation under the individual constraints with the cost of non-cooperating companies. Therefore, we compare these costs by means of numerical experiments. The problem parameters $(A, (\lambda_i, h_i)_{i \in N})$ are randomly chosen as follows:

- $A \in \{50, 100, \ldots, 250\}$,
- $\lambda_i \in \{20, 25, \ldots, 40\}$, and
- $h_i \in \{2, 6, 10\}$, $i = 1, 2$.

These numbers are similar to those used in the numerical tests in [16]. We use Matlab to perform 1250 random selections of the problem parameters. The numerical results are shown in table 1. In all these test instances cooperation is preferred.

A	Cost effectiveness		
	average	minimum	maximum
50	0.87	0.81	0.94
100	0.87	0.80	0.96
150	0.86	0.80	0.96
200	0.86	0.79	0.96
250	0.87	0.79	0.96

Table 1: The effect of cooperation for two companies. The cost effectiveness is the optimal joint cost divided by the total optimal cost for non-cooperating companies.

Proposition 12 *Cooperation under the individual constraints leads to lower expected joint costs than for non-cooperating companies. Also, cooperation with the individual optimal quantities leads to lower costs than for non-cooperating companies,* $K(Q_1^{nc}, Q_2^{nc}) \leq K_1(Q_1^{nc}) + K_2(Q_2^{nc})$.

Some other observations from the test instances are as follows. First, the cost function K is convex. Hence, it has a unique minimum. Second, the expected cycle time under cooperation is smaller than any cycle time of the non-cooperating companies. That is, a joint order for replenishment is placed more often than any individual order. Third, under cooperation the procurement cost per time unit is smaller than under individual optimization. Under cooperation the firms pay A per cycle instead of $2A$. On the other hand, the expected cycle time is smaller. Apparently the reduction in procurement cost dominates the decrease in cycle time. Finally, if the optimal order quantity under cooperation is the same as the individual optimal order quantity then the holding cost of firm j is larger under cooperation then under individual optimization. Under cooperation for each sample path of demands firm j has a weakly larger inventory position with probability 1. This causes a larger average inventory for firm j, so larger holding costs. On the other hand, with positive probability company $k \neq j$ ends the cycle. Then the cycle time is lower than under individual optimization. The total effect of the larger average inventory position and the lower cycle time is not clear beforehand. Apparently, the first effect dominates the latter.

5.2 Two identical companies

In this section we show that for cooperation under the individual constraints the joint cost for two *identical* companies has a simple expression. This allows us to analyse for which

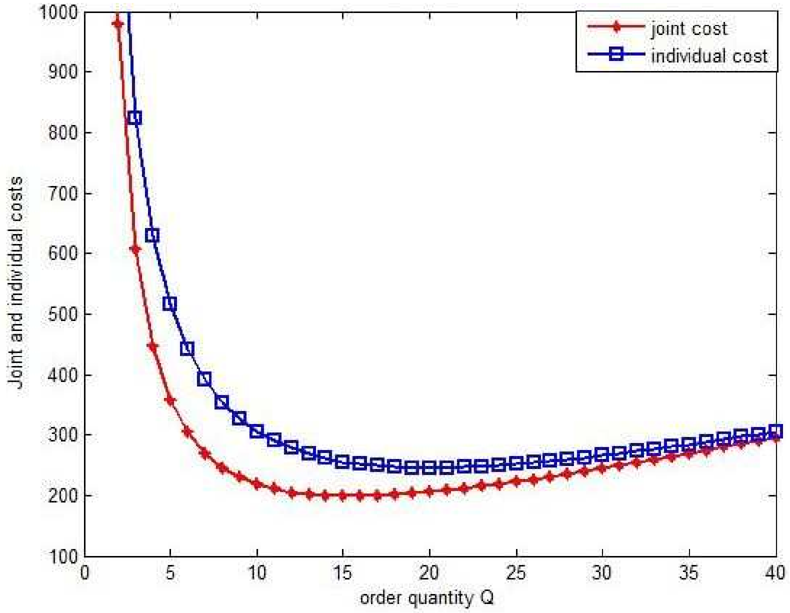

Figure 1: The joint cost $K(Q,Q)$ and total individual cost $K_1(Q)+K_2(Q)$ as a function of the order quantity Q for $A=20$, $\lambda=60$, and $h=6$.

values of the procurement cost A cooperation is beneficial.

Consider two identical companies, that is, the companies have equal parameters for holding cost, demand rate, and quantities. For ease of notation we denote these parameters without subscripts by h, λ and Q, respectively. Because the companies are identical, the proportion of demand for a company is $(p = 1-p =)$ 1/2. The expected joint cost is as follows.

Lemma 13 *In case of cooperation under individual constraints, the expected joint cost per time unit for two* identical *companies equals*

$$K(Q,Q) = \frac{A\lambda/Q + hQ}{1-\binom{2Q}{Q}\left(\frac{1}{2}\right)^{2Q}}$$

for order quantities (Q,Q).

The proof of this lemma is in appendix A. In figure 1 the joint cost $K(Q,Q)$ is compared to the total individual cost $K_1(Q)+K_2(Q)$ for parameter values $A=20$, $\lambda=60$, and $h=6$. Both cost functions are convex in the order quantity Q. The optimal joint cost (198.7) is lower than the individual optimal cost (246); the same relation holds for the optimal order quantity (15 and 20, respectively).

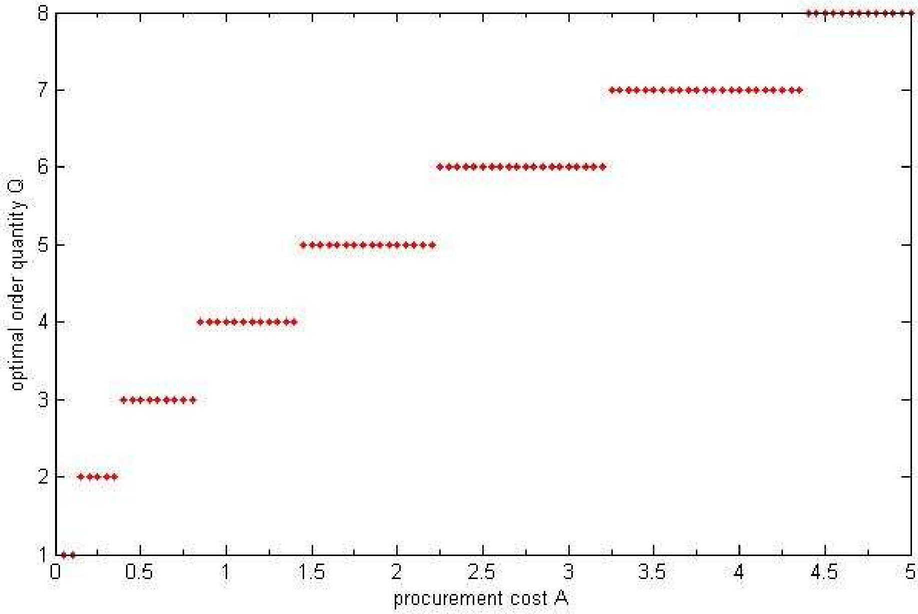

Figure 2: The optimal order quantity as a function of the procurement cost A for $\lambda = 60$, and $h = 6$.

Given the explicit formulation of the joint cost, the optimal order quantities (Q^*, Q^*) may be calculated. Figure 2 shows how the optimal order quantity depends on the procurement cost A for $\lambda = 60$, and $h = 6$. We may approximate the optimal order quantities Q^*. By Stirling's approximation $\binom{2Q}{Q}\left(\frac{1}{2}\right)^{2Q} \approx \frac{1}{\sqrt{\pi Q}}$. Thus, the joint cost $K(Q,Q)$ are approximately equal to $(A\lambda/Q + hQ)/(1 - 1/\sqrt{\pi Q})$. Taking the derivative with respect to Q leads to

$$2h\sqrt{\pi}Q^{5/2} - 3hQ^2 - 2A\lambda\sqrt{\pi}Q^{1/2} + A\lambda = 0.$$

This equation characterizes the minimum order quantity Q of the approximate cost function.

Now the question arises when cooperation is beneficial. In other words, when is the optimal joint cost lower than optimal total individual cost, $K(Q^*, Q^*) < K_1(Q^{nc}) + K_2(Q^{nc})$?

In this section we determine the value of the procurement cost A such that the companies have the same total cost under cooperation and non-cooperation. As we saw in table 1, if procurement is expensive, then cooperation is beneficial. Now if the procurement cost A decreases, then replenishing inventory becomes cheaper. Hence, the companies replenish more often and the order quantity per replenishment decreases for both individual firms and cooperating firms. If the procurement cost A is low enough then the order quantities are equal to one unit, $Q^* = 1$ and $Q^{nc} = 1$. In this case, cooperation has the same cost as non-cooperation: $K(1,1) = 2(A\lambda + h) = K_1(1) + K_2(1)$. This is also true for even lower values of A. Therefore, we determine the largest value of the procurement cost A such that the cost of cooperation is equal to the total individual cost. We call this the *switch value* of A, and denote it by $\bar{A}$.

Theorem 14 *For two identical companies cooperation is beneficial if and only if $A > \bar{A} = h/\lambda$.*

Proof. By definition of the switch value, cooperation under the individual constraints has lower cost than individual optimization if and only if $A > \bar{A}$. To determine the switch value, we observe the following. If the procurement cost A is slightly larger than $\bar{A}$, then the optimal order quantities increase to 2 (because they are integer valued). In other words, the switch value is the smallest value of A such that the optimal order quantities are equal to 2, and the costs of cooperation and non-cooperation are the same. By lemma 13, $K(2,2) = \frac{4}{5}(A\lambda + 4h)$. Also, $K_1(2) + K_2(2) = A\lambda + 3h$. The solution of $K(2,2) = K_1(2) + K_2(2)$ is $A = h/\lambda$. $\square$

6 Cost allocation

Proposition 12 states that cooperation under the individual constraints reduces costs. In this section we investigate how to allocate the joint costs among the companies. For this, we use cooperative game theory as a tool.

We start by introducing cooperative games. The two-company replenishment game is a cost game (N, c). $N = \{1, 2\}$ is the player set consisting of the two companies. A coalition U of players is a nonempty subset of N. The cost function c assigns to any coalition of players a cost; by convention $c(\emptyset) = 0$. In this game, the cost of firm i is the minimal cost of cost function $K_i(Q_i)$, as defined in (1); $c(\{i\}) = K_i(Q_i^{nc})$ for $i \in N$. The cost of coalition N is the minimal cost of the joint cost function $K(Q_1, Q_2)$ as defined in lemma 9; $c(N) = \min_{(Q_1,Q_2)} K(Q_1, Q_2)$. A game (N, c) is called *concave* if $c(U_1 \cup U_2) + c(U_1 \cap U_2) \leq c(U_1) + c(U_2)$ for any coalitions U_1, U_2.

There are several ways to allocate the joint cost $c(N)$. A first requirement is to use an allocation in the core $C(N, c)$,

$$C(N,c) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = c(N),\ \sum_{i \in T} x_i \leq c(U) \text{ for all } U \right\},$$

if this set is nonempty. If an element of the core is proposed as an allocation of the joint cost $c(N)$ then all coalitions U pay a quantity that is at most equal to their cost $c(U)$. Hence, no coalition wants to deviate from the cooperation within coalition N. Therefore, such an allocation is called a *stable* allocation.

Cooperation is always better than individual optimization, $c(N) < c(\{1\}) + c(\{2\})$, according to proposition 12. Consequently, the two-firm replenishment game is concave and the core

$$C(\{1,2\},c) = \{x \in \mathbb{R}^2 : x_1 + x_2 = c(N);\ x_i \leq c(\{i\}),\ i = 1,2\}$$

is a nonempty set. Since the core contains more than one allocation, a natural question that arises is which core element to select.

We consider two specific allocations. The first one is the Shapley value [13]. It is defined as follows. Let σ be a permutation of the players with player $\sigma(k)$ in position k. The marginal vector $m^\sigma(c)$ is a vector that assigns to each player its marginal contribution to the cost for the permutation σ:

$$m^\sigma_{\sigma(i)}(c) = \begin{cases} c(\{\sigma(i)\}), & i = 1, \\ c(\{\sigma(1),\sigma(2),\ldots,\sigma(i)\}) - c(\{\sigma(1),\sigma(2),\ldots,\sigma(i-1)\}), & i > 1. \end{cases}$$

The *Shapley value* $\phi(c)$ is an allocation of the joint cost $c(N)$ such that each player pays its average marginal contribution to the costs:

$$\phi(c) = \frac{1}{|N|!} \sum_\sigma m^\sigma(c),$$

where $|N|$ is the number of players. If the game is concave then the Shapley value belongs to the core of the game [14].

Second, the *distribution rule* $\delta(c)$ is a cost allocation designed for inventory cost games. This rule is an extension of the distribution rule for deterministic inventory cost games [12]. The latter rule belongs to the core and can be reached through a population monotonic allocation scheme, which is a scheme that determines a cost allocation for any coalition such that the cost allocated to a player is decreasing with the coalition. The distribution rule consists of two parts. Let (Q_1^N, Q_2^N) be the optimal pair of order quantities under cooperation. The first part is the distribution of the joint average order costs $A(\lambda_1 + \lambda_2)/G(Q_1^N, Q_2^N)$ among the firms proportional to the square of the individual optimal order costs $(A\lambda_i/Q_i^{nc})^2$. The second part is the individual holding cost of each firm pays as

Companies	1	2	1, 2
Optimal cost	287.86	405.00	549.95
Shapley value	216.40	333.55	
Distribution rule	197.98	351.97	

Table 2: The cost allocations for two companies with problem parameters $A = 200$, $\lambda_1 = 20$, $\lambda_2 = 40$, $h_1 = 10$, $h_2 = 10$.

experienced under cooperation. The distribution rule allocates

$$\begin{aligned}\delta_1(c) &= \frac{(A\lambda_1/Q_1^{nc})^2}{\sum_{j\in N}(A\lambda_i/Q_i^{nc})^2}\frac{A(\lambda_1+\lambda_2)}{G(Q_1^N,Q_2^N)} \\ &+ \frac{Q_2^N}{G(Q_1^N,Q_2^N)}\sum_{z_1=0}^{Q_1^N-1} h_1(Q_1^N - z_1/2)\binom{z_1+Q_2^N}{z_1}p^{z_1}(1-p)^{Q_2^N} \\ &+ \frac{Q_1^N}{G(Q_1^N,Q_2^N)}\sum_{z_2=0}^{Q_2^N-1} h_1(Q_1^N+1)/2\binom{Q_1^N+z_2}{z_2}p^{Q_1^N}(1-p)^{z_2}\end{aligned}$$

to company 1, and allocates

$$\begin{aligned}\delta_2(c) &= \frac{(A\lambda_2/Q_2^{nc})^2}{\sum_{j\in N}(A\lambda_i/Q_i^{nc})^2}\frac{A(\lambda_1+\lambda_2)}{G(Q_1^N,Q_2^N)} \\ &+ \frac{Q_2^N}{G(Q_1^N,Q_2^N)}\sum_{z_1=0}^{Q_1^N-1} h_2(Q_2^N+1)/2\binom{z_1+Q_2^N}{z_1}p^{z_1}(1-p)^{Q_2^N} \\ &+ \frac{Q_1^N}{G(Q_1^N,Q_2^N)}\sum_{z_2=0}^{Q_2^N-1} h_2(Q_2^N - z_2/2)\binom{Q_1^N+z_2}{z_2}p^{Q_1^N}(1-p)^{z_2}\end{aligned}$$

to company 2.

Both cost allocations are stable allocations according to the numerical calculations; see table 2. In all test instances, the replenishment game has a nonempty core and is concave. Then the Shapley value is a stable allocation because it belongs to the core of the game [14]. Also the distribution rule belongs to the core of the game and is therefore stable.

Proposition 15 *The Shapley value $\phi(c)$, and the distribution rule $\delta(c)$ are stable cost allocations in the two-company replenishment game.*

7 Joint replenishment under individual constraints for multiple companies

In this section we extend our analysis of the cooperation strategy under the individual constraints to a finite set N of companies. All the proofs of this section are in appendix A.

The cooperation strategy is the same as in section 5: all companies reorder as soon as one of them reaches its re-order level. The state space is

$$S = \{n = (n_i)_{i \in N} : 1 \leq n_i \leq Q_i,\ i \in N\},$$

and the corresponding equilibrium distribution is stated in the lemma below, with $Q = (Q_i)_{i \in N}$.

Lemma 16 *The equilibrium distribution π is a truncated multinomial distribution:*

$$\pi(n) = \lim_{t \to \infty} P(Z^t = n) = \frac{1}{G(Q)} \rho(n),$$

for states $n \in S$, where $G(Q) = \sum_{n \in S} \rho(n)$ is the normalising constant,

$$\rho(n) = \frac{(\sum_{i \in N}(Q_i - n_i))!}{\Pi_{i \in N}(Q_i - n_i)!} \Pi_{i \in N} p_i^{Q_i - n_i},$$

and $p_i = \lambda_i / \sum_{j \in N} \lambda_j$ is the proportion of demand for company i.

We proceed by calculating the probability that a company ends a cycle, and the expected cycle time.

Lemma 17 *The probability that company i ends a cycle is*

$$\sum_{j \neq i} \sum_{z_j = 0}^{Q_j - 1} \frac{(Q_i - 1 + \sum_{j \neq i} z_j)!}{(Q_i - 1)! \Pi_{j \neq i} z_j!} p_i^{Q_i} \Pi_{j \neq i} p_j^{z_j},$$

with $z_j = Q_j - n_j$. The expected cycle time is $\mathbb{E}T = G(Q) / \sum_{j \in N} \lambda_j$.

We can now derive the cost rate for the cooperating companies.

Lemma 18 *The average joint cost per time unit for a set N of companies equals*

$$K_N(Q) = A \frac{\sum_{j \in N} \lambda_j}{G(Q)} + \sum_{i \in N} \frac{Q_i}{G(Q)} \cdot$$
$$\cdot \sum_{j \neq i} \sum_{z_j = 0}^{Q_j - 1} \left[h_i (Q_i + 1)/2 + \sum_{j \neq i} h_j (Q_j - z_j/2) \right] \frac{(Q_i + \sum_{j \neq i} z_j)!}{Q_i! \Pi_{j \neq i} z_j!} p_i^{Q_i} \Pi_{j \neq i} p_j^{z_j}.$$

A	Cost effectiveness		
	average	minimum	maximum
50	0.72	0.69	0.74
100	0.70	0.66	0.73
150	0.69	0.65	0.72
200	0.68	0.65	0.72
250	0.68	0.65	0.71

Table 3: The effect of cooperation for three companies. The cost ratio is the optimal joint cost divided by the total optimal cost for non-cooperating companies.

We proceed by analysing whether cooperation is worthwhile. For ease of calculation we numerically investigate situations with three companies. The problem parameters $(A, (\lambda_i, h_i)_{i\in N})$ of the inventory situations that we consider are randomly chosen as follows:

- $A \in \{50, 100, \ldots, 250\}$,
- $\lambda_i \in \{20, 25, \ldots, 40\}$, and
- $h_i \in \{2, 6, 10\}$, $i = 1, 2, 3$.

These parameters are the same as used in section 5. We use Matlab to perform 1250 random selections of the parameters and calculate the corresponding optimal costs. In all test instances, cooperation is preferred; see table 3.

Proposition 19 *Cooperation under the individual constraints for three companies leads to lower joint costs than for non-cooperating companies.*

According to this proposition, the companies have an incentive to cooperate. The next question that comes to mind is how to allocate the joint costs. As in section 6, we use cooperative game theory to answer that question. We extend the two-company replenishment game to a game with multiple companies.

A multi-company replenishment game is the cost game (N, c) with finite player set N. Let $\{Q_i^U\}_{i\in U}$ be the optimal order quantities that minimize the cost $K_U(\{Q_i\}_{i\in U})$. Notice that $Q_i^{\{i\}} = Q_i^{nc}$ for a single-company coalition $U = \{i\}$. The cost function c assigns to a coalition U of companies its minimal cost $c(U) = K_U(\{Q_i^U\}_{i\in U})$. The definition of the distribution rule in section 6 can readily be extended to multi-company replenishment games.

Companies	1	2	3	1, 2	1, 3	2, 3	1, 2, 3
Optimal cost	358.57	174.21	276.87	424.78	497.58	350.95	553.26
Shapley value	265.51	100.01	187.74				
Distribution rule	291.30	79.23	182.73				

Table 4: The cost allocations for three companies with problem parameters $A = 250$, $\lambda_1 = 25$, $\lambda_2 = 30$, $\lambda_3 = 25$, $h_1 = 10$, $h_2 = 2$, $h_3 = 6$.

In all test instances the three-company replenishment game is a concave game. The Shapley value and the distribution rule are both stable allocations; see table 4.

Proposition 20 *For replenishment situations with three companies, the Shapley value and the distribution rule are stable cost allocations.*

These stability results are similar to those for deterministic inventory games in [12].

8 Conclusions

In this paper we study cooperation among multiple companies to control their joint inventories. These inventories are reviewed continuously, and the companies face Poisson demand. The paper starts with situations with two companies. A first cooperation strategy prescribes the companies to place a joint order for replenishment of their stocks when their joint inventory position reaches a certain reorder level. This strategy leads to a simple expression of the expected joint cost. This cost, however, exceeds the joint cost of non-cooperating companies.

The second cooperation strategy prescribes the firms to reorder as soon as a company reaches its reorder level. The formulation of the expected joint cost is more complex than for the first cooperation strategy, but numerical experiments show that the joint cost are lower than for non-cooperating companies. For two identical companies we characterise the values of the procurement cost such that cooperation is beneficial. The numerical experiments also show that the distribution rule and the Shapley value are stable allocations of the joint cost.

Finally, we extend the second cooperation strategy to multi-company replenishment situations. For ease of calculations, the numerical experiments consider situations with three companies. Also here, the numerical results show that cooperation reduces costs, and that the distribution rule and the Shapley value are stable cost allocations.

In future research we like to extend our model to include a positive lead time, back-orders, and to study other suitable cost allocations.

A Proofs

Proof of lemma 1. Substitution into the global balance equations (2)-(5) shows that

$$\begin{aligned}\tilde{\pi}(n_1, n_2) &= \lim_{t\to\infty} P\big(\big(Z_1^t, Z_2^t\big) = (n_1, n_2)\big)\\ &= \tilde{G}^{-1}\begin{pmatrix} Q_1 - n_1 + Q_2 - n_2 \\ Q_1 - n_1 \end{pmatrix} p^{Q_1-n_1}(1-p)^{Q_2-n_2},\end{aligned}$$

is the unique equilibrium distribution. The normalising constant $\tilde{G}^{-1}$ is readily computed as

$$\begin{aligned}\tilde{G} &= \sum_{(n_1,n_2)\in\tilde{S}} \begin{pmatrix} Q_1 - n_1 + Q_2 - n_1 - n_2 \\ Q_1 - n_1 \end{pmatrix} p^{Q_1-n_1}(1-p)^{Q_2-n_2}\\ &= \sum_{z_2=0}^{Q_2-1}\sum_{z_1=0}^{Q_2-1-z_2} \begin{pmatrix} z_1 + z_2 \\ z_1 \end{pmatrix} p^{z_1}(1-p)^{z_2} = \sum_{n=0}^{Q_2-1}\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} p^{k}(1-p)^{n-k} = Q_2.\end{aligned}$$

□

Proof of lemma 3. We start by calculating the probability that a demand for one of the companies ends the cycle. Let $P(n_1, n_2)$ be the probability that the system reaches state (n_1, n_2) in a cycle. The regeneration point is reached for sure — $P(Q_1, Q_2) = 1$ — and $P(n_1, n_2) = pP(n_1 + 1, n_2) + (1 - p)P(n_1, n_2 + 1)$. Notice that this equation resembles equation (2), hence the solution is

$$P(n_1, n_2) = \rho(n_1, n_2) = \begin{pmatrix} Q_1 + Q_2 - n_1 - n_2 \\ Q_1 - n_1 \end{pmatrix} p^{Q_1-n_1}(1-p)^{Q_2-n_2}.$$

Next define the probability $P_1(Q_1 - x, x)$ that a cycle terminates due to a demand for company 1 in state $(Q_1 - x + 1, x)$. Then

$$P_1(Q_1 - x, x) = pP(Q_1 - x + 1, x) = \begin{pmatrix} Q_2 - 1 \\ x - 1 \end{pmatrix} p^{x}(1-p)^{Q_2-x}$$

for $x = 1, \ldots, Q_2$. Similarly, the probability $P_2(Q_1 - y, y)$ that a cycle terminates due to a demand for company 2 at state $(Q_1 - y, y + 1)$ is

$$P_2(Q_1 - y, y) = (1 - p)P(Q_1 - y, y + 1) = \begin{pmatrix} Q_2 - 1 \\ y \end{pmatrix} p^{y}(1-p)^{Q_2-y}$$

for $y = 0, \dots, Q_2 - 1$. The expected holding cost incurred during a cycle that is terminated by company 1 in state $(Q_1 - x, x)$, $x = 1, \dots, Q_2$, is the expected cost times the cycle length:

$$[h_1(2Q_1 - x + 1)/2 + h_2(Q_2 + x)/2]\,\frac{Q_2}{\lambda_1 + \lambda_2}.$$

Similarly, the expected holding cost during a cycle that is terminated by company 2 in state $(Q_1 - y, y)$, $y = 0, \dots, Q_2 - 1$, is

$$[h_1(2Q_1 - y)/2 + h_2(Q_2 + y + 1)/2]\,\frac{Q_2}{\lambda_1 + \lambda_2}.$$

Thus the expected holding cost during a cycle terminated by company 1 is

$$\begin{aligned}
&\sum_{x=1}^{Q_2} [h_1(2Q_1 - x + 1)/2 + h_2(Q_2 + x)/2]\,\frac{Q_2}{\lambda_1 + \lambda_2} P_1(Q_1 - x, x) \\
&= \frac{Q_2}{\lambda_1 + \lambda_2} \sum_{x=1}^{Q_2} \left[h_1 \left(Q_1 - \frac{x-1}{2} \right) + h_2 \frac{Q_2 + x}{2} \right] \begin{pmatrix} Q_2 - 1 \\ x - 1 \end{pmatrix} p^x (1-p)^{Q_2 - x} \\
&= \frac{Q_2}{\lambda_1 + \lambda_2} p \sum_{y=0}^{Q_2 - 1} \left[h_1 \left(Q_1 - \frac{y}{2} \right) + h_2 \frac{Q_2 + y + 1}{2} \right] \begin{pmatrix} Q_2 - 1 \\ y \end{pmatrix} p^y (1-p)^{Q_2 - 1 - y}
\end{aligned}$$

Let Y be the random variable with realisations y in this expression. This variable Y is binomial distributed with $Q_2 - 1$ trials and probability p of success. Therefore we proceed:

$$\begin{aligned}
&= \frac{Q_2}{\lambda_1 + \lambda_2} p \mathbb{E} \left[h_1 \left(Q_1 - \frac{1}{2} Y \right) + h_2 \left(\frac{1}{2}(Q_2 + 1) + \frac{1}{2} Y \right) \right] \\
&= \frac{Q_2}{\lambda_1 + \lambda_2} p \left[h_1 \left(Q_1 - \frac{1}{2} p (Q_2 - 1) \right) + h_2 \left(\frac{1}{2}(Q_2 + 1) + \frac{1}{2} p (Q_2 - 1) \right) \right].
\end{aligned}$$

In a similar fashion we derive the expected holding cost during a cycle terminated by company 2.

$$\begin{aligned}
&\sum_{y=0}^{Q_2 - 1} \left[h_1 \left(2Q_1 - y \right)/2 + h_2 \left(Q_2 + y + 1 \right)/2 \right] \frac{Q_2}{\lambda_1 + \lambda_2} P_2(Q_1 - y, y) \\
&= \frac{Q_2}{\lambda_1 + \lambda_2} (1-p) \left[h_1 \left(Q_1 - \frac{1}{2} p (Q_2 - 1) \right) + h_2 \left(\frac{1}{2}(Q_2 + 1) + \frac{1}{2} p (Q_2 - 1) \right) \right].
\end{aligned}$$

Summarizing, the expected procurement and holding cost per cycle are

$$
\begin{aligned}
&A + \frac{Q_2}{\lambda_1+\lambda_2} p \left[h_1 \left(Q_1 - \frac{1}{2} p (Q_2 - 1) \right) + h_2 \left(\frac{1}{2}(Q_2+1) + \frac{1}{2} p (Q_2 - 1) \right) \right] \\
&+ \frac{Q_2}{\lambda_1+\lambda_2} (1-p) \left[h_1 \left(Q_1 - \frac{1}{2} p (Q_2 - 1) \right) + h_2 \left(\frac{1}{2}(Q_2+1) + \frac{1}{2} p (Q_2 - 1) \right) \right] \\
&= A + \frac{Q_2}{\lambda_1+\lambda_2} \left[h_1 \left(Q_1 - \frac{1}{2} p (Q_2 - 1) \right) + h_2 \left(\frac{1}{2}(Q_2+1) + \frac{1}{2} p (Q_2 - 1) \right) \right].
\end{aligned}
$$

Dividing this by the expected cycle time results in the expected joint cost per time unit. □

Proof of theorem 5. Let Q^s be the optimal order quantity for cooperation under the sum constraint. Then

$$
\begin{aligned}
\tilde{K}(Q^s, Q^s) &= A\frac{\lambda_1+\lambda_2}{Q^s} + \frac{1}{2} h_1 \left((2-p)Q^s + p\right) + \frac{1}{2} h_2 \left((1+p)Q^s + 1 - p\right) \\
&> A\frac{\lambda_1+\lambda_2}{Q^s} + \frac{1}{2}(h_1+h_2)(Q^s+1) \\
&= K_1(Q^s) + K_2(Q^s),
\end{aligned}
$$

where the inequality follows from $Q^s > 1$; if $Q^s = 1$ then equality holds. Thus if both companies use order quantity Q^s then the total individual costs do not exceed the optimal cost of cooperation under the sum constraint. But then the same conclusion holds for the total optimal individual cost. □

Proof of lemma 8. The cycle ends because of a demand for company 1 or company 2. If company 2 ends the cycle then the expected cycle length is the expected time it takes for $Q_1 - n_1 + Q_2$ demands to occur times the probability that company 2 triggers the replenishment order while company 1 still has n_1 items on stock. A similar interpretation holds for the case that company 1 ends the cycle. This leads to

$$\begin{aligned}
\mathbb{E}T &= \sum_{n_1=1}^{Q_1} \frac{Q_1 - n_1 + Q_2}{\lambda_1 + \lambda_2} P(n_1, 0) + \sum_{n_2=1}^{Q_2} \frac{Q_1 + Q_2 - n_2}{\lambda_1 + \lambda_2} P(0, n_2) \\
&= \sum_{z_1=0}^{Q_1-1} \frac{z_1 + Q_2}{\lambda_1 + \lambda_2} \binom{z_1 + Q_2 - 1}{z_1} p^{z_1} (1-p)^{Q_2} \\
&\quad + \sum_{z_2=0}^{Q_2-1} \frac{Q_1 + z_2}{\lambda_1 + \lambda_2} \binom{Q_1 - 1 + z_2}{z_2} p^{Q_1} (1-p)^{z_2} \\
&= \frac{Q_2}{(\lambda_1 + \lambda_2)(1-p)} \sum_{z_1=0}^{Q_1-1} \binom{z_1 + Q_2}{z_1} p^{z_1} (1-p)^{Q_2+1} \\
&\quad + \frac{Q_1}{(\lambda_1 + \lambda_2)p} \sum_{z_2=0}^{Q_2-1} \binom{Q_1 + z_2}{z_2} p^{Q_1+1} (1-p)^{z_2} \\
&= \frac{Q_2}{\lambda_2} I_{1-p}(Q_2 + 1, Q_1) + \frac{Q_1}{\lambda_1} I_p(Q_1 + 1, Q_2).
\end{aligned}$$

The second result follows from lemma 7. □

Proof of lemma 9. The expected holding cost incurred during a cycle are the holding cost per time unit times the cycle length

$$[h_1(Q_1 + n_1)/2 + h_2(Q_2 + 1)/2] \, \frac{Q_1 - n_1 + Q_2}{\lambda_1 + \lambda_2}$$

if firm 2 ends the cycle while the inventory position of firm 1 is n_1, and

$$[h_1(Q_1 + 1)/2 + h_2(Q_2 + n_2)/2] \, \frac{Q_1 + Q_2 - n_2}{\lambda_1 + \lambda_2}$$

if firm 1 ends the cycle while the inventory position of firm 2 is n_2. Then the expected joint procurement and holding costs per cycle under cooperation are

$$\begin{aligned}
&A + \sum_{n_1=1}^{Q_1} [h_1(Q_1 + n_1)/2 + h_2(Q_2 + 1)/2] \, \frac{Q_1 - n_1 + Q_2}{\lambda_1 + \lambda_2} P(n_1, 0) \\
&+ \sum_{n_2=1}^{Q_2} [h_1(Q_1 + 1)/2 + h_2(Q_2 + n_2)/2] \, \frac{Q_1 + Q_2 - n_2}{\lambda_1 + \lambda_2} P(0, n_2) \\
&= A + \frac{Q_2}{\lambda_1 + \lambda_2} \sum_{z_1=0}^{Q_1-1} [h_1(Q_1 - z_1/2) + h_2(Q_2 + 1)/2] \binom{z_1 + Q_2}{z_1} p^{z_1} (1-p)^{Q_2} \\
&\quad + \frac{Q_1}{\lambda_1 + \lambda_2} \sum_{z_2=0}^{Q_2-1} [h_1(Q_1 + 1)/2 + h_2(Q_2 - z_2/2)] \binom{Q_1 + z_2}{z_2} p^{Q_1} (1-p)^{z_2}
\end{aligned}$$

The average cost per time unit K is obtained by dividing this cost by the expected cycle time in lemma 7. $\square$

Proof of lemma 13. According to lemma 8, $G(Q,Q) = 4QI_{\frac{1}{2}}(Q+1,Q)$. By lemma 9,

$$\begin{aligned} &K(Q,Q) \\ &= \frac{A\lambda/Q}{2I_{\frac{1}{2}}(Q+1,Q)} + \frac{1}{2I_{\frac{1}{2}}(Q+1,Q)} \sum_{z=0}^{Q-1} [h(3Q+1)/2 - hz/2] \binom{z+Q}{z} \left(\frac{1}{2}\right)^{z+Q}. \end{aligned}$$

The summation in this expression reduces to:

$$\begin{aligned} &\sum_{z=0}^{Q-1} [h(3Q+1)/2 - hz/2] \binom{z+Q}{z} \left(\frac{1}{2}\right)^{z+Q} \\ &= h(3Q+1) \sum_{z=0}^{Q-1} \binom{z+Q}{z} \left(\frac{1}{2}\right)^{z+Q+1} - h(Q+1) \sum_{z=1}^{Q-1} \binom{z+Q}{Q+1} \left(\frac{1}{2}\right)^{z+Q+1} \\ &= h(3Q+1) I_{\frac{1}{2}}(Q+1,Q) - h(Q+1) I_{\frac{1}{2}}(Q+2,Q-1). \end{aligned}$$

Therefore,

$$K(Q,Q) = \frac{A\lambda/Q}{2I_{\frac{1}{2}}(Q+1,Q)} + \frac{h}{2}(3Q+1) - h(Q+1) \frac{I_{\frac{1}{2}}(Q+2,Q-1)}{2I_{\frac{1}{2}}(Q+1,Q)}.$$

Using the equality $I_{\frac{1}{2}}(Q+2,Q-1) = I_{\frac{1}{2}}(Q+1,Q) - \binom{2Q}{Q+1}\left(\frac{1}{2}\right)^{2Q}$ [1, (26.5.15)], we obtain

$$K(Q,Q) = \frac{A\lambda/Q}{2I_{\frac{1}{2}}(Q+1,Q)} + hQ + h(Q+1) \frac{\binom{2Q}{Q+1}\left(\frac{1}{2}\right)^{2Q}}{2I_{\frac{1}{2}}(Q+1,Q)}.$$

Finally, because $h(Q+1)\binom{2Q}{Q+1}\left(\frac{1}{2}\right)^{2Q} = hQ\binom{2Q-1}{Q}\left(\frac{1}{2}\right)^{2Q-1}$, and $I_{\frac{1}{2}}(Q+1,Q) = 1 - I_{\frac{1}{2}}(Q,Q+1) = \frac{1}{2} - \binom{2Q}{Q}\left(\frac{1}{2}\right)^{2Q+1}$, we conclude

$$K(Q,Q) = \frac{A\lambda/Q}{2I_{\frac{1}{2}}(Q+1,Q)} + hQ\left(1 + \frac{\binom{2Q-1}{Q}\left(\frac{1}{2}\right)^{2Q-1}}{2I_{\frac{1}{2}}(Q+1,Q)}\right) = \frac{A\lambda/Q + hQ}{1 - \binom{2Q}{Q}\left(\frac{1}{2}\right)^{2Q}}.$$

$\square$

Proof of lemma 16. The state space S contains three types of states. First, inner states are states in $S_I = \{n \in S : 1 \le n_i < Q_i \text{ for all } i \in N\}$; all firms have experienced demand. Let $e^i \in \mathbb{R}^N$ be the vector defined by $e^i_i = 1$ and $e^i_j = 0$ for $j \neq i$, and let $\pi(n)$ denote the steady state probability for state n. The flow balance equations for the inner states are

$$\sum_{i \in N} \lambda_i \pi(n) = \sum_{i \in N} \lambda_i \pi(n + e^i) \qquad (9)$$

for all $n \in S_I$. Second, $S_B = \{n \in S : \text{for some } j \in N\ n_j = Q_j;\ n_i < Q_i,\ i \neq j\}$ is the set of boundary states related to situations in which demand has occurred for all but one firm. The flow balance equations for these states are

$$\sum_{i \in N} \lambda_i \pi(n) = \sum_{i \neq j} \lambda_i \pi(n + e^i) \tag{10}$$

if $n_j = Q_j$. The final type of state is the regeneration point $S_R = \{Q\}$ that is reached each time the companies jointly order. For this state the flows should satisfy

$$\sum_{i \in N} \lambda_i \pi(Q) = \sum_{i \in N} \lambda_i \sum_{n \in S : n_i = 1} \pi(n). \tag{11}$$

Substitution of the equilibrium distribution into the global balance equations (9)-(11) yields the result immediately. □

Proof of lemma 17. First, let $P(n)$ denote the probability that the system reaches state n in a cycle. Then $P(n) = \rho(n)$. The derivation of this result is similar to that for the two-company case. Let n^{1i} denote a state n with $n_i = 1$. The probability that firm i terminates a cycle is $\sum_{n^{1i} \in S} p_i P(n^{1i})$, the probability that the state is n^{1i} and a demand for firm i arrives.

$$\begin{aligned}
\sum_{n^{1i} \in S} p_i P(n^{1i}) &= \sum_{j \neq i} \sum_{n_j = 1}^{Q_j} p_i \frac{(Q_i - 1 + \sum_{j \neq i}(Q_j - n_j))!}{(Q_i - 1)! \Pi_{j \neq i}(Q_j - n_j)!} p_i^{Q_i - 1} \Pi_{j \neq i} p_j^{Q_j - n_j} \\
&= \sum_{j \neq i} \sum_{n_j = 1}^{Q_j} \frac{(Q_i - 1 + \sum_{j \neq i}(Q_j - n_j))!}{(Q_i - 1)! \Pi_{j \neq i}(Q_j - n_j)!} p_i^{Q_i} \Pi_{j \neq i} p_j^{Q_j - n_j} \\
&= \sum_{j \neq i} \sum_{z_j = 0}^{Q_j - 1} \frac{(Q_i - 1 + \sum_{j \neq i} z_j)!}{(Q_i - 1)! \Pi_{j \neq i} z_j!} p_i^{Q_i} \Pi_{j \neq i} p_j^{z_j},
\end{aligned}$$

where we use $z_j = Q_j - n_j$ in the last equality.

Second, we calculate the expected cycle time.

$$\begin{aligned}
\mathbb{E}T &= \int_0^\infty P(T>t)dt \;=\; \int_0^\infty \Pi_{i\in N} P(T_i>t)dt \\
&= \int_0^\infty \Pi_{i\in N}\left(\sum_{k_i=0}^{Q_i-1}\frac{(\lambda_i t)^{k_i}}{k_i!}e^{-\lambda_i t}\right)dt \\
&= \frac{1}{\sum_{j\in N}\lambda_j}\int_0^\infty \Pi_{i\in N}\sum_{k_i=0}^{Q_i-1}\frac{(p_i u)^{k_i}}{k_i!}e^{-u}du \\
&= \frac{1}{\sum_{j\in N}\lambda_j}\sum_{i\in N}\sum_{k_i=0}^{Q_i-1}\frac{(\sum_{j\in N}k_j)!}{\Pi_{j\in N}k_j!}\Pi_{j\in N}p_j^{k_j}\int_0^\infty\frac{u^{\sum_{j\in N}k_j}}{(\sum_{j\in N}k_j)!}e^{-u}du \\
&= \frac{1}{\sum_{j\in N}\lambda_j}\sum_{i\in N}\sum_{k_i=0}^{Q_i-1}\frac{(\sum_{j\in N}k_j)!}{\Pi_{j\in N}k_j!}\Pi_{j\in N}p_j^{k_j} \;=\; \frac{G(Q)}{\sum_{j\in N}\lambda_j},
\end{aligned}$$

where we use $u = t\sum_{j\in N}\lambda_j$ in the fourth equality. □

Proof of lemma 18. Assume company i terminates a cycle. Then this incurs expected holding costs (average cost per time unit times expected cycle length)

$$\left[h_i(Q_i+1)/2+\sum_{j\neq i}h_j(Q_j+n_j)/2\right]\frac{Q_i+\sum_{j\neq i}(Q_j-n_j)}{\sum_{j\in N}\lambda_j}$$

for the companies during the cycle. Thus the expected joint cost per cycle are

$$\begin{aligned}
&A+\sum_{i\in N}\sum_{n^{1i}\in S}\left[h_i(Q_i+1)/2+\sum_{j\neq i}h_j(Q_j+n_j)/2\right]\frac{Q_i+\sum_{j\neq i}(Q_j-n_j)}{\sum_{j\in N}\lambda_j}p_iP(n^{1i}) \\
&= A+\sum_{i\in N}\sum_{j\neq i}\sum_{z_j=0}^{Q_j-1}\left[h_i(Q_i+1)/2+\sum_{j\neq i}h_j(Q_j-z_j/2)\right]\cdot \\
&\qquad\qquad \cdot\frac{Q_i+\sum_{j\neq i}z_j}{\sum_{j\in N}\lambda_j}\frac{(Q_i-1+\sum_{j\neq i}z_j)!}{(Q_i-1)!\Pi_{j\neq i}z_j!}p_i^{Q_i}\Pi_{j\neq i}p_j^{z_j} \\
&= A+\sum_{i\in N}\sum_{j\neq i}\sum_{z_j=0}^{Q_j-1}\left[h_i(Q_i+1)/2+\sum_{j\neq i}h_j(Q_j-z_j/2)\right]\cdot \\
&\qquad\qquad \cdot\frac{Q_i}{\sum_{j\in N}\lambda_j}\frac{(Q_i+\sum_{j\neq i}z_j)!}{Q_i!\Pi_{j\neq i}z_j!}p_i^{Q_i}\Pi_{j\neq i}p_j^{z_j}.
\end{aligned}$$

Divide by the expected cycle time $\mathbb{E}T = G(Q)/\sum_{j\in N}\lambda_j$ to obtain the result. □

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